1.1-Part 1

In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. The functions \( f(x) = x + \sqrt{2 - x} \) and \( g(u) = u + \sqrt{2 - u} \) give exactly the same output values for every input value, so \( f \) and \( g \) are equal.

2. \( f(x) = \frac{x^2 - x}{x - 1} = \frac{x(x - 1)}{x - 1} = x \) for \( x \neq 1 \), so \( f \) and \( g \) [where \( g(x) = x \)] are not equal because \( f(1) \) is undefined and \( g(1) = 1 \).

3. (a) The point \((-1, -2)\) is on the graph of \( f \), so \( f(-1) = -2 \).

   (b) When \( x = 2 \), \( y \) is about 2.8, so \( f(2) \approx 2.8 \).

   (c) \( f(x) = 2 \) is equivalent to \( y = 2 \). When \( y = 2 \), we have \( x = -3 \) and \( x = 1 \).

   (d) Reasonable estimates for \( x \) when \( y = 0 \) are \( x = -2.5 \) and \( x = 0.3 \).

   (e) The domain of \( f \) consists of all \( x \)-values on the graph of \( f \). For this function, the domain is \( -3 \leq x \leq 3 \), or \([-3, 3]\).

   The range of \( f \) consists of all \( y \)-values on the graph of \( f \). For this function, the range is \( -2 \leq y \leq 3 \), or \([-2, 3]\).

   (f) As \( x \) increases from \(-1\) to \( 3 \), \( y \) increases from \(-2\) to \( 3 \). Thus, \( f \) is increasing on the interval \([-1, 3]\).

4. (a) The point \((-4, -2)\) is on the graph of \( f \), so \( f(-4) = -2 \). The point \((3, 4)\) is on the graph of \( g \), so \( g(3) = 4 \).

   (b) We are looking for the values of \( x \) for which the \( y \)-values are equal. The \( y \)-values for \( f \) and \( g \) are equal at the points \((-2, 1)\) and \((2, 2)\), so the desired values of \( x \) are \(-2\) and \( 2 \).

   (c) \( f(x) = -1 \) is equivalent to \( y = -1 \). When \( y = -1 \), we have \( x = -3 \) and \( x = 4 \).

   (d) As \( x \) increases from \( 0 \) to \( 4 \), \( y \) decreases from \( 3 \) to \(-1 \). Thus, \( f \) is decreasing on the interval \([0, 4]\).

   (e) The domain of \( f \) consists of all \( x \)-values on the graph of \( f \). For this function, the domain is \(-4 \leq x \leq 4 \), or \([-4, 4]\).

   The range of \( f \) consists of all \( y \)-values on the graph of \( f \). For this function, the range is \(-2 \leq y \leq 3 \), or \([-2, 3]\).

   (f) The domain of \( g \) is \([-4, 3]\) and the range is \([0.5, 4]\).

5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is \([-2, 2]\) and the range is \([-1, 2]\).

7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is \([-3, 2]\) and the range is \([-3, -2) \cup [-1, 3]\).
8. No, the curve is not the graph of a function since for $x = 0, \pm 1, \text{ and } \pm 2$, there are infinitely many points on the curve.

9. The person’s weight increased to about 64 kg at age 20 and stayed fairly steady for 10 years. The person’s weight dropped to about 52 kg for the next 5 years, then increased rapidly to about 68 kg. The next 30 years saw a gradual increase to 76 kg.
   Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

10. First, the tub was filled with water to a height of 38 cm. Then a person got into the tub, raising the water level to 50 cm. At around 12 minutes, the person stood up in the tub but then immediately sat down. Finally, at around 17 minutes, the person got out of the tub, and then drained the water.

11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.

12. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)

13. Of course, this graph depends strongly on the geographical location!
14. The value of the car decreases fairly rapidly initially, then somewhat less rapidly.

![Graph showing value decreasing over time.]

15. As the price increases, the amount sold decreases.

![Graph showing amount decreasing over price.]

16. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.

![Graph showing temperature fluctuating over time.]

17. Height of grass

![Graph showing growth pattern over time.]

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18. (a) \[ x(t) \]

(b) \[ y(t) \]

(c) \[ \text{ground speed (km/h)} \]

(d) \[ \text{vertical velocity} \]

19. \[ f(x) = 3x^2 - x + 2. \]

\[ f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12 \]

\[ f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16. \]

\[ f(a) = 3a^2 - a + 2. \]

\[ f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2. \]

\[ f(a + 1) = 3(a + 1)^2 - (a + 1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4. \]

\[ 2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4. \]

\[ f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2. \]

\[ f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2. \]

\[ [f(a)]^2 = [3a^2 - a + 2]^2 = (3a^2 - a + 2)(3a^2 - a + 2) \]

\[ = 9a^4 - 3a^2 + 6a^2 - 3a^2 + a^2 - 2a + 6a^2 - 2a + 2 = 9a^4 + 6a^2 + 13a^2 - 4a + 4. \]

\[ f(a + h) = 3(a + h)^2 - (a + h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2. \]

20. A spherical balloon with radius \(r + 1\) has volume \(V(r + 1) = \frac{4}{3}\pi (r + 1)^3 = \frac{4}{3}\pi (r^3 + 3r^2 + 3r + 1)\). We wish to find the amount of air needed to inflate the balloon from a radius of \(r\) to \(r + 1\). Hence, we need to find the difference

\[ V(r + 1) - V(r) = \frac{4}{3}\pi (r^3 + 3r^2 + 3r + 1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (3r^2 + 3r + 1). \]

21. \( f(x) = 4 + 3x - x^2 \), so \( f(3 + h) = 4 + 3(3 + h) - (3 + h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2. \)

\[ \text{and} \frac{f(3 + h) - f(3)}{h} = \frac{(4 - 3h - h^2) - 4}{h} = \frac{h(-3 - h)}{h} = -3 - h. \]
22. \( f(x) = x^2 \), so \( f(a + h) = (a + h)^3 = a^3 + 3a^2h + 3ah^2 + h^3 \),

and \( \frac{f(a + h) - f(a)}{h} = \frac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} = \frac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2 \).

23. \( \frac{f(x) - f(a)}{x - a} = \frac{1}{x} - \frac{1}{a} = \frac{a - x}{xa(x - a)} = \frac{-1(x - a)}{xa(x - a)} = -\frac{1}{ax} \)

24. \( \frac{f(x) - f(1)}{x - 1} = \frac{x + 3 - 2}{x - 1} = \frac{x + 3 - 2(x + 1)}{x - 1} = \frac{x + 3 - 2x - 2}{(x + 1)(x - 1)} = \frac{-x + 1}{(x + 1)(x - 1)} = -\frac{1}{x + 1} \)

25. \( f(x) = (x + 4)/(x^2 - 9) \) is defined for all \( x \) except when \( 0 = x^2 - 9 \Rightarrow 0 = (x + 3)(x - 3) \Rightarrow x = -3 \) or \( 3 \), so the domain is \( \{x \in \mathbb{R} \mid x \neq -3, 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty) \).

26. \( f(x) = (2x^2 - 5)/(x^2 + x - 6) \) is defined for all \( x \) except when \( 0 = x^2 + x - 6 \Rightarrow 0 = (x + 3)(x - 2) \Rightarrow x = -3 \) or \( 2 \), so the domain is \( \{x \in \mathbb{R} \mid x \neq -3, 2\} = (-\infty, -3) \cup (-3, 2) \cup (2, \infty) \).

27. \( F(p) = \sqrt{2 - \sqrt{p}} \) is defined when \( p \geq 0 \) and \( 2 - \sqrt{p} \geq 0 \). Since \( 2 - \sqrt{p} \geq 0 \Rightarrow 2 \geq \sqrt{p} \Rightarrow \sqrt{p} \leq 2 \Rightarrow 0 \leq p \leq 4 \), the domain is \([0, 4]\).

28. \( g(t) = \sqrt{3 - t} - \sqrt{2 + t} \) is defined when \( 3 - t \geq 0 \Rightarrow t \leq 3 \) and \( 2 + t \geq 0 \Rightarrow t \geq -2 \). Thus, the domain is \(-2 \leq t \leq 3 \), or \([-2, 3]\).

29. \( h(x) = 1/\sqrt{x^2 - 5x} \) is defined when \( x^2 - 5x > 0 \Rightarrow x(x - 5) > 0 \). Note that \( x^2 - 5x \neq 0 \) since that would result in division by zero. The expression \( x(x - 5) \) is positive if \( x < 0 \) or \( x > 5 \). (See Review of Algebra at www.stewartcalculus.com for methods for solving inequalities.) Thus, the domain is \((-\infty, 0) \cup (5, \infty)\).

30. \( h(x) = \sqrt{4 - x^2} \). Now \( y = \sqrt{4 - x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 4 \), so the graph is the top half of a circle of radius 2 with center at the origin. The domain is \( \{x \mid 4 - x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2, 2] \). From the graph, the range is \( 0 \leq y \leq 2 \), or \([0, 2]\).
31. \( f(x) = 5 \) is defined for all real numbers, so the domain is \( \mathbb{R} \), or \((-\infty, \infty)\). The graph of \( f \) is a horizontal line with y-intercept 5.

\[ \begin{align*} \end{align*} \]

32. \( F(x) = x^2 - 2x + 1 = (x - 1)^2 \) is defined for all real numbers, so the domain is \( \mathbb{R} \), or \((-\infty, \infty)\). The graph of \( F \) is a parabola with vertex \((1, 0)\).

\[ \begin{align*} \end{align*} \]

33. \( f(t) = 2t + t^2 \) is defined for all real numbers, so the domain is \( \mathbb{R} \), or \((-\infty, \infty)\). The graph of \( f \) is a parabola opening upward since the coefficient of \( t^2 \) is positive. To find the \( t \)-intercepts, let \( y = 0 \) and solve for \( t \). \( 0 = 2t + t^2 = t(2 + t) \Rightarrow t = 0 \) or \( t = -2 \). The \( t \)-coordinate of the vertex is halfway between the \( t \)-intercepts, that is, at \( t = -1 \). Since \( f(-1) = 2(-1) + (-1)^2 = -2 + 1 = -1 \), the vertex is \((-1, -1)\).

\[ \begin{align*} \end{align*} \]

34. \( H(t) = \frac{4 - t^2}{2 - t} = \frac{(2 + t)(2 - t)}{2 - t} \), so for \( t \neq 2 \), \( H(t) = 2 + t \). The domain is \( \{ t \mid t \neq 2 \} \). So the graph of \( H \) is the same as the graph of the function \( f(t) = t + 2 \) (a line) except for the hole at \((2, 4)\).

\[ \begin{align*} \end{align*} \]

35. \( g(x) = \sqrt{x - 5} \) is defined when \( x - 5 \geq 0 \) or \( x \geq 5 \), so the domain is \([5, \infty)\).

Since \( y = \sqrt{x - 5} \Rightarrow y^2 = x - 5 \Rightarrow x = y^2 + 5 \), we see that \( g \) is the top half of a parabola.

\[ \begin{align*} \end{align*} \]

36. \( F(x) = |2x + 1| = \begin{cases} 2x + 1 & \text{if } 2x + 1 \geq 0 \\ -(2x + 1) & \text{if } 2x + 1 < 1 \end{cases} \]

\[ \begin{align*} \end{align*} \]

The domain is \( \mathbb{R} \), or \((-\infty, \infty)\).
37. \( G(x) = \frac{3x + |x|}{x} \). Since \(|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}\), we have

\[
G(x) = \begin{cases} 
\frac{3x + x}{x} & \text{if } x > 0 \\
\frac{3x - x}{x} & \text{if } x < 0
\end{cases}
\]

\[
= \begin{cases} 
\frac{4x}{x} & \text{if } x > 0 \\
\frac{2x}{x} & \text{if } x < 0
\end{cases}
\]

\[
= \begin{cases} 
4 & \text{if } x > 0 \\
2 & \text{if } x < 0
\end{cases}
\]

Note that \( G \) is not defined for \( x = 0 \). The domain is \((-\infty, 0) \cup (0, \infty)\).

38. \( g(x) = \frac{|x|}{x^2} \). Since \(|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}\), we have

\[
g(x) = \begin{cases} 
\frac{x}{x^2} & \text{if } x > 0 \\
\frac{-x}{x^2} & \text{if } x < 0
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{x} & \text{if } x > 0 \\
\frac{1}{-x} & \text{if } x < 0
\end{cases}
\]

Note that \( g \) is not defined for \( x = 0 \). The domain is \((-\infty, 0) \cup (0, \infty)\).

39. \( f(x) = \begin{cases} 
x + 2 & \text{if } x < 0 \\
1 - x & \text{if } x \geq 0
\end{cases}\)

The domain is \( \mathbb{R} \).

40. \( f(x) = \begin{cases} 
3 - \frac{3}{2}x & \text{if } x \leq 2 \\
2x - 5 & \text{if } x > 2
\end{cases}\)

The domain is \( \mathbb{R} \).
41. \( f(x) = \begin{cases} 
  x + 2 & \text{if } x \leq -1 \\
  x^2 & \text{if } x > -1
\end{cases} \)

Note that for \( x = -1 \), both \( x + 2 \) and \( x^2 \) are equal to 1.

The domain is \( \mathbb{R} \).

42. \( f(x) = \begin{cases} 
  -1 & \text{if } x \leq -1 \\
  3x + 2 & \text{if } -1 < x < 1 \\
  7 - 2x & \text{if } x \geq 1
\end{cases} \)

The domain is \( \mathbb{R} \).

43. Recall that the slope \( m \) of a line between the two points \((x_1, y_1)\) and \((x_2, y_2)\) is \( m = \frac{y_2 - y_1}{x_2 - x_1} \) and an equation of the line connecting those two points is \( y - y_1 = m(x - x_1) \). The slope of the line segment joining the points \((1, -3)\) and \((5, 7)\) is \( \frac{7 - (-3)}{5 - 1} = \frac{5}{2} \), so an equation is \( y - (-3) = \frac{5}{2}(x - 1) \). The function is \( f(x) = \frac{5}{2}x - \frac{11}{2}, 1 \leq x \leq 5 \).

44. The slope of this line segment is \( \frac{3 - (-2)}{6 - (-3)} = \frac{5}{9} \), so an equation is \( y + 2 = \frac{5}{9}(x + 3) \).

The function is \( f(x) = \frac{5}{9}x - \frac{17}{3}, -3 \leq x \leq 6 \).

45. We need to solve the given equation for \( y \). \( x + (y - 1)^2 = 0 \), so \( (y - 1)^2 = -x \) \( \Rightarrow y - 1 = \pm \sqrt{-x} \) \( \Rightarrow y = 1 \pm \sqrt{-x} \). The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want \( f(x) = 1 - \sqrt{-x} \). Note that the domain is \( x \leq 0 \).

46. \( x^2 + (y - 2)^2 = 4 \), so \( (y - 2)^2 = 4 - x^2 \) \( \Rightarrow y - 2 = \pm \sqrt{4 - x^2} \) \( \Rightarrow y = 2 \pm \sqrt{4 - x^2} \). The top half is given by the function \( f(x) = 2 + \sqrt{4 - x^2}, -2 \leq x \leq 2 \).
47. Let the length and width of the rectangle be \( L \) and \( W \). Then the perimeter is \( 2L + 2W = 20 \) and the area is \( A = LW \).

Solving the first equation for \( W \) in terms of \( L \) gives \( W = \frac{20 - 2L}{2} = 10 - L \). Thus, \( A(L) = L(10 - L) = 10L - L^2 \). Since lengths are positive, the domain of \( A \) is \( 0 < L < 10 \). If we further restrict \( L \) to be larger than \( W \), then \( 5 < L < 10 \) would be the domain.

48. Let the length and width of the rectangle be \( L \) and \( W \). Then the area is \( LW = 16 \), so that \( W = 16/L \). The perimeter is \( P = 2L + 2W \), so \( P(L) = 2L + 2(16/L) = 2L + 32/L \), and the domain of \( P \) is \( L > 0 \), since lengths must be positive quantities. If we further restrict \( L \) to be larger than \( W \), then \( L > 4 \) would be the domain.

49. Let the length of a side of the equilateral triangle be \( x \). Then by the Pythagorean Theorem, the height \( y \) of the triangle satisfies \( y^2 + \left(\frac{1}{2}x\right)^2 = x^2 \), so that \( y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2 \) and \( y = \frac{\sqrt{3}}{2}x \). Using the formula for the area \( A \) of a triangle, \( A = \frac{1}{2}(\text{base})(\text{height}) \), we obtain \( A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2 \), with domain \( x > 0 \).
50. Let the volume of the cube be \( V \) and the length of an edge be \( L \). Then \( V = L^3 \) so \( L = \sqrt[3]{V} \), and the surface area is
\[
S(V) = 6L^2 = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}, \quad \text{with domain } V > 0.
\]

51. Let each side of the base of the box have length \( x \), and let the height of the box be \( h \). Since the volume is 2, we know that \( 2 = hx^2 \), so that \( h = 2/x^2 \), and the surface area is \( S = x^2 + 4xh \). Thus, \( S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x) \), with domain \( x > 0 \).

52. We can summarize the monthly cost with a piecewise defined function.
\[
C(x) = \begin{cases} 
25 & \text{if } 0 \leq x \leq 400 \\
35 + 0.10(x - 400) & \text{if } x > 400 
\end{cases}
\]

53. (a) \( R(\%) \)

\[
\begin{array}{c|c|c|c|c|c}
\hline
x & 0 & 10,000 & 20,000 & 30,000 & 25000 \\
\hline
T & 0 & 10 & 15 & 25 & 100 \\
\hline
\end{array}
\]

(b) On \$14,000, tax is assessed on \$4000, and \( 10\%(\$4000) = \$400 \).
On \$26,000, tax is assessed on \$16,000, and \( 10\%(\$10,000) + 15\%(\$6000) = \$1000 + \$900 = \$1900 \).

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of \( T \) is a line segment from \((10,000, 0)\) to \((20,000, 1000)\).
The tax on \$30,000 is \$2500, so the graph of \( T \) for \( x > 20,000 \) is the ray with initial point \((20,000, 1000)\) that passes through \((30,000, 2500)\).

54. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

55. \( f \) is an odd function because its graph is symmetric about the origin. \( g \) is an even function because its graph is symmetric with respect to the \( y \)-axis.

56. \( f \) is not an even function since it is not symmetric with respect to the \( y \)-axis. \( f \) is not an odd function since it is not symmetric about the origin. Hence, \( f \) is \textit{neither} even nor odd. \( g \) is an even function because its graph is symmetric with respect to the \( y \)-axis.
57. (a) Because an even function is symmetric with respect to the $y$-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

58. (a) If $f$ is even, we get the rest of the graph by reflecting about the $y$-axis.

(b) If $f$ is odd, we get the rest of the graph by rotating $180^\circ$ about the origin.

59. $f(x) = \frac{x}{x^2 + 1}$.

$$f(-x) = \frac{-x}{(-x)^2 + 1} = \frac{-x}{x^2 + 1} = -\frac{x}{x^2 + 1} = -f(x).$$

So $f$ is an odd function.

60. $f(x) = \frac{x^2}{x^4 + 1}$.

$$f(-x) = \frac{(-x)^2}{(-x)^4 + 1} = \frac{x^2}{x^4 + 1} = f(x).$$

So $f$ is an even function.
61. \( f(x) = \frac{x}{x + 1}, \) so \( f(-x) = \frac{-x}{-x + 1} = \frac{x}{x - 1}. \)

Since this is neither \( f(x) \) nor \( -f(x) \), the function \( f \) is neither even nor odd.

62. \( f(x) = x \left| x \right|. \)

\[
f(-x) = (-x) \left| -x \right| = (-x) \left| x \right| = -(x \left| x \right|)
= -f(x)
\]

So \( f \) is an odd function.

63. \( f(x) = 1 + 3x^2 - x^4. \)

\[
f(-x) = 1 + 3(-x)^2 - (-x)^4 = 1 + 3x^2 - x^4 = f(x)
\]

So \( f \) is an even function.
64. \( f(x) = 1 + 3x^2 - x^5 \), so

\[
\begin{align*}
  f(-x) &= 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5) \\
        &= 1 - 3x^3 + x^5
\end{align*}
\]

Since this is neither \( f(x) \) nor \( -f(x) \), the function \( f \) is
neither even nor odd.

65. (i) If \( f \) and \( g \) are both even functions, then \( f(-x) = f(x) \) and \( g(-x) = g(x) \). Now

\[
(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x),
\]
so \( f + g \) is an even function.

(ii) If \( f \) and \( g \) are both odd functions, then \( f(-x) = -f(x) \) and \( g(-x) = -g(x) \). Now

\[
(f + g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -(f(x) + g(x)) = -(f + g)(x),
\]
so \( f + g \) is an odd function.

(iii) If \( f \) is an even function and \( g \) is an odd function, then \( (f + g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x) \),
which is not \( f + g(x) \) nor \( -(f + g)(x) \), so \( f + g \) is neither even nor odd. (Exception: if \( f \) is the zero function, then
\( f + g \) will be odd. If \( g \) is the zero function, then \( f + g \) will be even.)

66. (i) If \( f \) and \( g \) are both even functions, then \( f(-x) = f(x) \) and \( g(-x) = g(x) \). Now

\[
(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x),
\]
so \( fg \) is an even function.

(ii) If \( f \) and \( g \) are both odd functions, then \( f(-x) = -f(x) \) and \( g(-x) = -g(x) \). Now

\[
(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x),
\]
so \( fg \) is an even function.

(iii) If \( f \) is an even function and \( g \) is an odd function, then

\[
(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x),
\]
so \( fg \) is an odd function.
1. (a) An equation for the family of linear functions with slope 2
    is \( y = f(x) = 2x + b \), where \( b \) is the \( y \)-intercept.

(b) \( f(2) = 1 \) means that the point \((2, 1)\) is on the graph of \( f \). We can use the
    point-slope form of a line to obtain an equation for the family of linear
    functions through the point \((2, 1)\). \( y - 1 = m(x - 2) \), which is equivalent
    to \( y = mx + (1 - 2m) \) in slope-intercept form.

(c) To belong to both families, an equation must have slope \( m = 2 \), so the equation in part (b), \( y = mx + (1 - 2m) \),
    becomes \( y = 2x - 3 \). It is the only function that belongs to both families.

2. All members of the family of linear functions \( f(x) = 1 + m(x + 3) \) have
    graphs that are lines passing through the point \((-3, 1)\).

3. All members of the family of linear functions \( f(x) = c - x \) have graphs
    that are lines with slope \(-1\). The \( y \)-intercept is \( c \).
4. The vertex of the parabola on the left is \((3, 0)\), so an equation is \(y = a(x - 3)^2 + 0\). Since the point \((4, 2)\) is on the parabola, we’ll substitute 4 for \(x\) and 2 for \(y\) to find \(a\). \[2 = a(4 - 3)^2 \Rightarrow a = 2, \text{ so an equation is } f(x) = 2(x - 3)^2.\]

The \(y\)-intercept of the parabola on the right is \((0, 1)\), so an equation is \(y = ax^2 + bx + 1\). Since the points \((-2, 2)\) and \((1, -2.5)\) are on the parabola, we’ll substitute \(-2\) for \(x\) and 2 for \(y\) as well as 1 for \(x\) and \(-2.5\) for \(y\) to obtain two equations with the unknowns \(a\) and \(b\).

\[
\begin{align*}
(-2, 2): & \quad 2 = 4a - 2b + 1 \Rightarrow 4a - 2b = 1 \\
(1, -2.5): & \quad -2.5 = a + b + 1 \Rightarrow a + b = -3.5
\end{align*}
\]

\(2 \cdot (2) + (1) \text{ gives us } 6a = -6 \Rightarrow a = -1. \text{ From (2), } -1 + b = -3.5 \Rightarrow b = -2.5, \text{ so an equation is } g(x) = -x^2 - 2.5x + 1.\)

5. Since \(f(-1) = f(0) = f(2) = 0, \) \(f\) has zeros of \(-1, 0, \) and \(2, \) so an equation for \(f\) is \(f(x) = a[x - (-1)][x - 0][x - 2],\) or \(f(x) = ax(x + 1)(x - 2). \) Because \(f(1) = 6, \) we’ll substitute 1 for \(x\) and 6 for \(f(x)\).

\[6 = a(1)(2)(-1) \Rightarrow -2a = 6 \Rightarrow a = -3, \text{ so an equation for } f \text{ is } f(x) = -3x(x + 1)(x - 2).\]

6. \(a)\) For \(T = 0.02t + 8.50, \) the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of 0.02 °C per year. The \(T\)-intercept is 8.50, which represents the average surface temperature in °C in the year 1900. 
\(b)\) \(t = 2100 - 1900 = 200 \Rightarrow T = 0.02(200) + 8.50 = 12.50 \text{ °C}.\)

7. \(a)\) \(D = 200, \) so \(c = 0.0417D(a + 1) = 0.0417(200)(a + 1) = 8.34a + 8.34. \) The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

\(b)\) For a newborn, \(a = 0, \) so \(c = 8.34 \text{ mg.}\)

8. \(a)\) \(y\) \(200\) \(100\) \(0\) \(10\) \(20\) \(30\) \(40\) \(50\) \(60\) \(x\)

\(b)\) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented decreases by 4. The \(y\)-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The \(x\)-intercept of 50 is the smallest rental fee that results in no spaces rented.
9. (a) \[ F = \frac{9}{5}C + 32 \] (b) The slope of \( \frac{9}{5} \) means that \( F \) increases \( \frac{9}{5} \) degrees for each increase of 1°C. (Equivalently, \( F \) increases by 9 when \( C \) increases by 5 and \( F \) decreases by 9 when \( C \) decreases by 5.) The \( F \)-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

10. (a) Let \( d \) = distance traveled (in km) and \( t \) = time elapsed (in hours). At \( t = 0 \), \( d = 0 \) and at \( t = 4 - 2 = 2 \) h, \( d = 210 \). Thus, we have two points: (0, 0) and (2, 210), so \( m = \frac{210 - 0}{2 - 0} = 105 \) and so \( d = 105t \).
   (c) The slope is 105 and represents the car's speed in km/h.

11. (a) Using \( N \) in place of \( x \) and \( T \) in place of \( y \), we find the slope to be \( \frac{T_2 - T_1}{N_2 - N_1} = \frac{29 - 20}{180 - 112} = \frac{9}{68} \). So a linear equation is \( T - 29 = \frac{9}{68} (N - 180) \) \( \iff \) \( T = \frac{9}{68} N + \frac{88}{17} \).
   (b) The slope of \( \frac{9}{68} \) means that the temperature in Celsius degrees increases nine sixty-eighths as rapidly as the number of cricket chirps per minute. Said differently, each increase of 68 cricket chirps per minute corresponds to an increase of 9°C.
   (c) When \( N = 150 \), the temperature is given approximately by \( T = \frac{9}{68} (150) + \frac{88}{17} \approx 25^\circ C \).

12. (a) Let \( x \) denote the number of chairs produced in one day and \( y \) the associated cost. Using the points (100, 2200) and (300, 4800), we get the slope \( \frac{4800 - 2200}{300 - 100} = \frac{2600}{200} = 13 \). So \( y - 2200 = 13(x - 100) \) \( \iff \) \( y = 13x + 900 \).
   (b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.
   (c) The \( y \)-intercept is 900 and it represents the fixed daily costs of operating the factory.

13. (a) We are given \( \frac{\text{change in pressure}}{\text{1 meter change in depth}} = \frac{0.10}{1} = 0.10 \). Using \( P \) for pressure and \( d \) for depth with the point \((d, P) = (0, 1.05)\), we have the slope-intercept form of the line, \( P = 0.10d + 1.05 \).
   (b) When \( P = 7 \), then \( 7 = 0.10d + 1.05 \) \( \iff \) \( 0.10d = 5.95 \) \( \iff \) \( d = \frac{5.95}{0.10} = 59.5 \) meters. Thus, the pressure is 7 kg/cm² at a depth of 59.5 meters.
14. (a) Using $d$ in place of $x$ and $C$ in place of $y$, we find the slope to be \[ \frac{C_2 - C_1}{d_2 - d_1} = \frac{460 - 380}{1280 - 768} = \frac{80}{512} = \frac{5}{32}. \]

So a linear equation is $C - 460 = \frac{5}{32} (d - 1280)$ \[ \Rightarrow \quad C = \frac{5}{32} d - 200 \Rightarrow \quad C = \frac{5}{32} d + 260. \]

(b) Letting $d = 2400$ we get $C = \frac{5}{32} (2400) + 260 = 635$.

The cost of driving 2400 km is $635.

(c) \[ \text{The slope of the line represents the cost per \text{ km} \ (about \$0.16).} \]

(d) The $y$-intercept represents the fixed cost, $260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional kilometer driven is a constant.

15. If $x$ is the original distance from the source, then the illumination is $f(x) = kx^{-2} = k/x^2$. Moving halfway to the lamp gives us an illumination of $f\left(\frac{1}{2}x\right) = k(\frac{1}{2}x)^{-2} = k(2/x)^2 = 4(k/x^2)$, so the light is 4 times as bright.

16. (a) If $A = 60$, then $S = 0.7A^{0.3} \approx 2.39$, so you would expect to find 2 species of bats in that cave.

(b) $S = 4 \Rightarrow 4 = 0.7A^{0.3} \Rightarrow \frac{40}{7} = A^{3/10} \Rightarrow A = \left(\frac{40}{7}\right)^{10/3} \approx 333.6$, so we estimate the surface area of the cave to be 334 m$^2$.

17. (a) If the graph of $f$ is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of $f$ is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of $f$ is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of $f$ is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of $f$ is reflected about the $x$-axis, its equation becomes $y = -f(x)$.

(f) If the graph of $f$ is reflected about the $y$-axis, its equation becomes $y = f(-x)$.

(g) If the graph of $f$ is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of $f$ is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3} f(x)$. 
18. (a) To obtain the graph of \( y = f(x) + 8 \) from the graph of \( y = f(x) \), shift the graph 8 units upward.

(b) To obtain the graph of \( y = f(x + 8) \) from the graph of \( y = f(x) \), shift the graph 8 units to the left.

(c) To obtain the graph of \( y = 8f(x) \) from the graph of \( y = f(x) \), stretch the graph vertically by a factor of 8.

(d) To obtain the graph of \( y = f(8x) \) from the graph of \( y = f(x) \), shrink the graph horizontally by a factor of 8.

(e) To obtain the graph of \( y = -f(x) - 1 \) from the graph of \( y = f(x) \), first reflect the graph about the \( x \)-axis, and then shift it 1 unit downward.

(f) To obtain the graph of \( y = 8f\left(\frac{1}{8}x\right) \) from the graph of \( y = f(x) \), stretch the graph horizontally and vertically by a factor of 8.

19. (a) (graph 3) The graph of \( f \) is shifted 4 units to the right and has equation \( y = f(x - 4) \).

(b) (graph 1) The graph of \( f \) is shifted 3 units upward and has equation \( y = f(x) + 3 \).

(c) (graph 4) The graph of \( f \) is shrunk vertically by a factor of 3 and has equation \( y = \frac{1}{3}f(x) \).

(d) (graph 5) The graph of \( f \) is shifted 4 units to the left and reflected about the \( x \)-axis. Its equation is \( y = -f(x + 4) \).

(e) (graph 2) The graph of \( f \) is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is \( y = 2f(x + 6) \).

20. (a) To graph \( y = f(x) - 2 \), we shift the graph of \( f \), 2 units downward. The point \((1, 2)\) on the graph of \( f \) corresponds to the point \((1, 2 - 2) = (1, 0)\).

(b) To graph \( y = f(x - 2) \), we shift the graph of \( f \), 2 units to the right. The point \((1, 2)\) on the graph of \( f \) corresponds to the point \((1 + 2, 2) = (3, 2)\).

(c) To graph \( y = -2f(x) \), we reflect the graph about the \( x \)-axis and stretch the graph vertically by a factor of 2. The point \((1, 2)\) on the graph of \( f \) corresponds to the point \((1, -2 \cdot 2) = (1, -4)\).

(d) To graph \( y = f\left(\frac{1}{2}x\right) + 1 \), we stretch the graph horizontally by a factor of 3 and shift it 1 unit upward. The point \((1, 2)\) on the graph of \( f \) corresponds to the point \((1 \cdot 3, 2 + 1) = (3, 3)\).
21. (a) To graph \( y = f(2x) \) we shrink the graph of \( f \) horizontally by a factor of 2.

\[
\begin{array}{c}
\text{Graph of } y = f(2x)
\end{array}
\]

The point \((4, -1)\) on the graph of \( f \) corresponds to the point \((\frac{1}{2} \cdot 4, -1) = (2, -1)\).

(b) To graph \( y = f\left(\frac{1}{2}x\right) \) we stretch the graph of \( f \) horizontally by a factor of 2.

\[
\begin{array}{c}
\text{Graph of } y = f\left(\frac{1}{2}x\right)
\end{array}
\]

The point \((4, -1)\) on the graph of \( f \) corresponds to the point \((2 \cdot 4, -1) = (8, -1)\).

(c) To graph \( y = f(-x) \) we reflect the graph of \( f \) about the \( y \)-axis.

\[
\begin{array}{c}
\text{Graph of } y = f(-x)
\end{array}
\]

The point \((4, -1)\) on the graph of \( f \) corresponds to the point \((-1 \cdot 4, -1) = (-4, -1)\).

(d) To graph \( y = -f(-x) \) we reflect the graph of \( f \) about the \( y \)-axis, then about the \( x \)-axis.

\[
\begin{array}{c}
\text{Graph of } y = -f(-x)
\end{array}
\]

The point \((4, -1)\) on the graph of \( f \) corresponds to the point \((-1 \cdot 4, -1 \cdot -1) = (-4, 1)\).

22. (a) The graph of \( y = 2 \sin x \) can be obtained from the graph of \( y = \sin x \) by stretching it vertically by a factor of 2.

\[
\begin{array}{c}
\text{Graph of } y = 2 \sin x
\end{array}
\]

(b) The graph of \( y = 1 + \sqrt{x} \) can be obtained from the graph of \( y = \sqrt{x} \) by shifting it upward 1 unit.

\[
\begin{array}{c}
\text{Graph of } y = 1 + \sqrt{x}
\end{array}
\]

23. \( y = -x^2 \): Start with the graph of \( y = x^3 \) and reflect about the \( x \)-axis. Note: Reflecting about the \( y \)-axis gives the same result since substituting \(-x\) for \( x \) gives us \( y = (-x)^2 = -x^2 \).

\[
\begin{array}{c}
\text{Graph of } y = x^3 \quad \text{and} \quad y = -x^3
\end{array}
\]
24. $y = (x - 1)^2$: Start with the graph of $y = x^2$ and shift 1 unit to the right.

\[ y = x^2 \quad \text{and} \quad y = (x - 1)^2 \]

25. $y = -\sqrt{x}$: Start with the graph of $y = \sqrt{x}$ and reflect about the x-axis.

\[ y = \sqrt{x} \quad \text{and} \quad y = -\sqrt{x} \]

26. $y = x^2 + 6x + 4 = (x^2 + 6x + 9) - 5 = (x + 3)^2 - 5$: Start with the graph of $y = x^2$, shift 3 units to the left, and then shift 5 units downward.

\[ y = x^2 \quad y = (x + 3)^2 \quad y = (x + 3)^2 - 5 \]

27. $y = \sqrt{x} - 2 - 1$: Start with the graph of $y = \sqrt{x}$, shift 2 units to the right, and then shift 1 unit downward.

\[ y = \sqrt{x} \quad y = \sqrt{x} - 2 \quad y = \sqrt{x} - 2 - 1 \]
28. \( y = 4 \sin 3x \): Start with the graph of \( y = \sin x \), compress horizontally by a factor of 3, and then stretch vertically by a factor of 4.

\[ y = \sin x \quad y = \sin 3x \quad y = 4 \sin 3x \]

29. \( y = \sin\left(\frac{x}{2}\right) \): Start with the graph of \( y = \sin x \) and stretch horizontally by a factor of 2.

\[ y = \sin x \quad y = \frac{1}{2} \sin 2x \quad y = \sin\left(\frac{x}{2}\right) \]

30. \( y = \frac{2}{x} - 2 \): Start with the graph of \( y = \frac{1}{x} \), stretch vertically by a factor of 2, and then shift 2 units downward.

\[ y = \frac{1}{x} \quad y = 2 \quad y = \frac{2}{x} - 2 \]

31. \( y = \frac{1}{2}(1 - \cos x) \): Start with the graph of \( y = \cos x \), reflect about the \( x \)-axis, shift 1 unit upward, and then shrink vertically by a factor of 2.

\[ y = \cos x \quad y = -\cos x \quad y = \frac{1}{2}(1 - \cos x) \]
32. \( y = 1 + \sqrt{x - 1} \): Start with the graph of \( y = \sqrt{x} \), shift 1 unit to the right, and then shift 1 unit upward.

33. \( y = 1 - 2x - x^2 = -(x^2 + 2x) + 1 = -(x^2 + 2x + 1) + 2 = -(x + 1)^2 + 2 \): Start with the graph of \( y = x^2 \), reflect about the \( x \)-axis, shift 1 unit to the left, and then shift 2 units upward.

34. \( y = |x| - 2 \): Start with the graph of \( y = |x| \) and shift 2 units downward.

35. \( y = \frac{2}{x+1} \). Start with the graph of \( y = \frac{1}{x} \), shift 1 unit to the left, and then stretch vertically by a factor of 2.
36. \( y = \frac{1}{4} \tan(x - \frac{\pi}{4}) \): Start with the graph of \( y = \tan x \), shift \( \frac{\pi}{4} \) units to the right, and then compress vertically by a factor of 4.

![Graphs of y = tan(x), y = tan(x - pi/4), and y = 1/4 * tan(x - pi/4)]

37. \( f(x) = x^2 + 2x^2; \ g(x) = 3x^2 - 1 \). \( D = \mathbb{R} \) for both \( f \) and \( g \).
   (a) \((f + g)(x) = (x^2 + 2x^2) + (3x^2 - 1) = x^2 + 5x^2 - 1, \ D = \mathbb{R} \).
   (b) \((f - g)(x) = (x^2 + 2x^2) - (3x^2 - 1) = x^2 - x^2 + 1, \ D = \mathbb{R} \).
   (c) \((fg)(x) = (x^2 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2, \ D = \mathbb{R} \).
   (d) \( \left( \frac{f}{g} \right)(x) = \frac{x^2 + 2x^2}{3x^2 - 1}, \ D = \left\{ x \mid x \neq \pm \frac{1}{\sqrt{3}} \right\} \) since \( 3x^2 - 1 \neq 0 \).

38. \( f(x) = \sqrt{1 + x}, \ D = [-1, \infty), \ g(x) = \sqrt{1 - x}, \ D = (-\infty, 1] \).
   (a) \((f + g)(x) = \sqrt{1 + x} + \sqrt{1 - x}, \ D = (-\infty, 1] \cap [-1, \infty) = [-1, 1] \).
   (b) \((f - g)(x) = \sqrt{1 + x} - \sqrt{1 - x}, \ D = [-1, 1] \).
   (c) \((fg)(x) = \sqrt{1 + x} \cdot \sqrt{1 - x} = \sqrt{1 - x^2}, \ D = [-1, 1] \).
   (d) \( \left( \frac{f}{g} \right)(x) = \frac{\sqrt{1 + x}}{\sqrt{1 - x}}, \ D = [-1, 1) \). We must exclude \( x = 1 \) since it would make \( \frac{f}{g} \) undefined.

39. \( f(x) = x^2 - 1, \ D = \mathbb{R}; \ g(x) = 2x + 1, \ D = \mathbb{R} \).
   (a) \((f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 - 1 = 4x^2 + 4x + 1 - 1 = 4x^2 + 4x, \ D = \mathbb{R} \).
   (b) \((g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 2(x^2 - 1) + 1 = (2x^2 - 2) + 1 = 2x^2 - 1, \ D = \mathbb{R} \).
   (c) \((f \circ f)(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1 = (x^4 - 2x^2 + 1) - 1 = x^4 - 2x^2, \ D = \mathbb{R} \).
   (d) \((g \circ g)(x) = g(g(x)) = g(2x + 1) = 2(2x + 1) + 1 = (4x + 2) + 1 = 4x + 3, \ D = \mathbb{R} \).
40. \( f(x) = 1 - x^3, D = \mathbb{R}; \ g(x) = 1/x, D = \{ x \mid x \neq 0 \}. \)
   (a) \((f \circ g)(x) = f(g(x)) = f(1/x) = 1 - (1/x)^3 = 1 - 1/x^3, D = \{ x \mid x \neq 0 \}.\)
   (b) \((g \circ f)(x) = g(f(x)) = g(1 - x^3) = 1/(1 - x^3), D = \{ x \mid 1 - x^3 \neq 0 \} = \{ x \mid x \neq 1 \}.\)
   (c) \((f \circ f)(x) = f(f(x)) = f(1 - x^3) = 1 - (1 - x^3)^3 \quad [= x^6 - 3x^4 + 3x^2], D = \mathbb{R}.\)
   (d) \((g \circ g)(x) = g(g(x)) = g(1/x) = 1/(1/x) = x, D = \{ x \mid x \neq 0 \}\) because 0 is not in the domain of \( g.\)

41. \( f(x) = 1 - 3x, \ g(x) = \cos x. \quad D = \mathbb{R} \) for both \( f \) and \( g \), and hence for their composites.
   (a) \((f \circ g)(x) = f(g(x)) = f(\cos x) = 1 - 3\cos x.\)
   (b) \((g \circ f)(x) = g(f(x)) = g(1 - 3x) = \cos(1 - 3x).\)
   (c) \((f \circ f)(x) = f(f(x)) = f(1 - 3x) = 1 - 3(1 - 3x) = 1 - 3 + 9x = 9x - 2.\)
   (d) \((g \circ g)(x) = g(g(x)) = g(\cos x) = \cos(\cos x) \quad [\text{Note that this is not } \cos x \cdot \cos x.]\)

42. \( f(x) = \sqrt{x}, \ D = [0, \infty); \ g(x) = \sqrt[3]{1-x}, \ D = \mathbb{R}. \)
   (a) \((f \circ g)(x) = f(g(x)) = f(\sqrt[3]{1-x}) = \sqrt{\sqrt[3]{1-x}} = \sqrt[3]{1-x}.\)
      The domain of \( f \circ g \) is \( \{ x \mid \sqrt[3]{1-x} \geq 0 \} = \{ x \mid 1 - x \geq 0 \} = \{ x \mid x \leq 1 \} = (-\infty, 1].\)
   (b) \((g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt[3]{1-\sqrt{x}}.\)
      The domain of \( g \circ f \) is \( \{ x \mid x \text{ is in the domain of } f \text{ and } f(x) \text{ is in the domain of } g \}. \text{ This is the domain of } f, \)
      that is, \([0, \infty)\).
   (c) \((f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt{x}. \text{ The domain of } f \circ f \text{ is } \{ x \mid x \geq 0 \text{ and } \sqrt{x} \geq 0 \} = [0, \infty).\)
   (d) \((g \circ g)(x) = g(g(x)) = g(\sqrt[3]{1-x}) = \sqrt[3]{1-\sqrt[3]{1-x}}, \text{ and the domain is } (-\infty, \infty).\)
43. \( f(x) = x + \frac{1}{x}, \ D = \{x \mid x \neq 0\}; \ g(x) = \frac{x + 1}{x + 2}, \ D = \{x \mid x \neq -2\} \)

(a) \((f \circ g)(x) = f(g(x)) = f\left(\frac{x + 1}{x + 2}\right) = \frac{x + 1}{x + 2} + \frac{1}{x + 1} + \frac{x + 1}{x + 2} = \frac{x + 1}{x + 2} + \frac{x + 2}{x + 1} = \frac{(x + 1)(x + 1) + (x + 2)(x + 2)}{(x + 2)(x + 1)} = \frac{x^2 + 2x + 1}{(x + 2)(x + 1)} + \frac{x^2 + 4x + 4}{(x + 2)(x + 1)} = \frac{2x^2 + 6x + 5}{(x + 2)(x + 1)}

Since \(g(x)\) is not defined for \(x = -2\) and \(f(g(x))\) is not defined for \(x = -2\) and \(x = -1\),
the domain of \((f \circ g)(x)\) is \(D = \{x \mid x \neq -2, -1\}\).

(b) \((g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x + 1} + \frac{1}{x + 1} = \frac{x^2 + 1 + x}{x^2 + 2x + 1} = \frac{x^2 + 1 + x}{(x + 1)^2}

Since \(f(x)\) is not defined for \(x = 0\) and \(g(f(x))\) is not defined for \(x = -1\),
the domain of \((g \circ f)(x)\) is \(D = \{x \mid x \neq -1, 0\}\).

(c) \((f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right) + \frac{1}{x + 1} = x + 1 + \frac{1}{x^2 + 1} = x + \frac{1}{x} + \frac{x}{x^2 + 1}

= \frac{x(x^2 + 1) + 1}{x(x^2 + 1)} = \frac{x^4 + x^3 + x^2 + 1 + x}{x(x^2 + 1)} = \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}, \ D = \{x \mid x \neq 0\}

(d) \((g \circ g)(x) = g(g(x)) = g\left(\frac{x + 1}{x + 2}\right) = \frac{x + 1}{x + 2} + \frac{1}{x + 2} = \frac{x + 1 + 1(x + 2)}{x + 2} = \frac{x + 1 + x + 2}{x + 2} = \frac{2x + 3}{3x + 5}

Since \(g(x)\) is not defined for \(x = -2\) and \(g(g(x))\) is not defined for \(x = -\frac{2}{3}\),
the domain of \((g \circ g)(x)\) is \(D = \{x \mid x \neq -2, -\frac{2}{3}\} \).
44. \( f(x) = \frac{x}{1 + x}, \quad D = \{x \mid x \neq -1\}; \quad g(x) = \sin 2x, \quad D = \mathbb{R}. \)

(a) \((f \circ g)(x) = f(g(x)) = f(\sin 2x) = \frac{\sin 2x}{1 + \sin 2x} \)

Domain: \(1 + \sin 2x \neq 0 \Rightarrow \sin 2x \neq -1 \Rightarrow 2x \neq \frac{3\pi}{2} + 2\pi n \Rightarrow x \neq \frac{3\pi}{4} + \pi n \quad [n \text{ an integer}]. \)

(b) \((g \circ f)(x) = g(f(x)) = g\left(\frac{x}{1 + x}\right) = \sin\left(\frac{2x}{1 + x}\right) \)

Domain: \(\{x \mid x \neq -1\} \)

(c) \((f \circ f)(x) = f(f(x)) = f\left(\frac{x}{1 + x}\right) = \frac{x}{1 + \frac{x}{1 + x}} = \frac{x}{1 + \frac{x}{1 + x}} \cdot \left(\frac{1 + x}{1 + x}\right) = \frac{x}{1 + x + x} = \frac{x}{2x + 1} \)

Since \(f(x)\) is not defined for \(x = -1\), and \(f(f(x))\) is not defined for \(x = -\frac{1}{2}\),

the domain of \((f \circ f)(x)\) is \(D = \{x \mid x \neq -1, -\frac{1}{2}\}\). 

(d) \((g \circ g)(g) = g(g(x)) = g(\sin 2x) = \sin(2 \sin 2x). \)

Domain: \(\mathbb{R}\)

45. \((f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2 + 2)) = f[(x^2 + 2)^2] \)

\[= f(x^6 + 4x^2 + 4) = \sqrt{x^6 + 4x^2 + 4} - 3 = \sqrt{x^6 + 4x^2 + 1} \]

46. \((f \circ g \circ h)(x) = f(g(h(x))) = f\left(\frac{\sqrt{x}}{\sqrt{x} - 1}\right) = \tan\left(\frac{\sqrt{x}}{\sqrt{x} - 1}\right) \)

47. Let \(g(x) = 2x + x^2\) and \(f(x) = x^4\). Then \((f \circ g)(x) = f(g(x)) = f(2x + x^2) = (2x + x^2)^4 = F(x)\).

48. Let \(g(x) = \sqrt{x}\) and \(f(x) = \sin x\). Then \((f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \sin(\sqrt{x}) = F(x)\).

49. Let \(g(t) = t^2\) and \(f(t) = \sec t \tan t\). Then \((f \circ g)(t) = f(g(t)) = f(t^2) = \sec(t^2) \tan(t^2) = u(t)\).

50. Let \(g(t) = \tan t\) and \(f(t) = \frac{t}{1 + t}\). Then \((f \circ g)(t) = f(g(t)) = f(\tan t) = \frac{\tan t}{1 + \tan t} = u(t)\).
51. Let $h(x) = x^2$, $g(x) = 3^x$, and $f(x) = 1 - x$. Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f\left(3^{x^2}\right) = 1 - 3^{x^2} = H(x).$$

52. Let $h(x) = |x|$, $g(x) = 2 + x$, and $f(x) = \sqrt[3]{x}$. Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(|x|)) = f(2 + |x|) = \sqrt[3]{2 + |x|} = H(x).$$

53. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sec \sqrt{x}) = (\sec \sqrt{x})^4 = \sec^4(\sqrt{x}) = H(x).$$

54. (a) $f(g(1)) = f(6) = 5$
   (b) $g(f(1)) = g(3) = 2$
   (c) $f(f(1)) = f(3) = 4$
   (d) $g(g(1)) = g(6) = 3$
   (e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$
   (f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

55. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of $g$. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of $f$.
   (b) $g(f(0)) = g(0) = 3$
   (c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$
   (d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of $g$ that has $x$-coordinate $6$.
   (e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$
   (f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

56. (a) The radius $r$ of the balloon is increasing at a rate of $2 \text{ cm/s}$, so $r(t) = (2 \text{ cm/s})(t \text{ s}) = 2t$ (in cm).
   (b) Using $V = \frac{4}{3}\pi r^3$, we get $V'(t) = V(r(t)) = V(2t) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^2$.
   The result, $V = \frac{32}{3}\pi t^2$, gives the volume of the balloon (in cm$^3$) as a function of time (in s).

57. (a) Using the relationship distance = rate \cdot time with the radius $r$ as the distance, we have $r(t) = 60t$.
   (b) $A = \pi r^2 \Rightarrow (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm$^2$) at any time $t$.

58. (a) $d = rt \Rightarrow d(t) = 350t$
   (b) There is a Pythagorean relationship involving the legs with lengths $d$ and $1$ and the hypotenuse with length $s$.
   $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.
   (c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$
59. (a) \[ H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \]

(b) \[ V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \] so \( V(t) = 120H(t) \).

Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of \( t = 0 \), we replace \( t \) with \( t - 5 \). Thus, the formula is \( V(t) = 240H(t-5) \).

60. (a) \[ R(t) = tH(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \]

(b) \[ V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases} \] so \( V(t) = 2tH(t), t \leq 60 \).

(c) \[ V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4(t-7) & \text{if } 7 \leq t \leq 32 \end{cases} \] so \( V(t) = 4(t-7)H(t-7), t \leq 32 \).

61. If \( f(x) = m_1x + b_1 \) and \( g(x) = m_2x + b_2 \), then

\[ (f \circ g)(x) = f(g(x)) = f(m_2x + b_2) = m_1(m_2x + b_2) + b_1 = m_1m_2x + m_1b_2 + b_1. \]

So \( f \circ g \) is a linear function with slope \( m_1m_2 \).

62. If \( A(x) = 1.04x \), then

\[ (A \circ A)(x) = A(A(x)) = A(1.04x) = 1.04(1.04x) = (1.04)^2x, \]

\( (A \circ A \circ A)(x) = A((A \circ A)(x)) = A((1.04)^2x) = 1.04(1.04)^2x = (1.04)^3x \), and

\( (A \circ A \circ A \circ A)(x) = A((A \circ A \circ A)(x)) = A((1.04)^3x) = 1.04(1.04)^3x \), etc.

These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose \( n \) copies of \( A \), we get the formula \( A \circ A \circ \cdots \circ A = (1.04)^n x \).
63. (a) By examining the variable terms in $g$ and $h$, we deduce that we must square $g$ to get the terms $4x^2$ and $4x$ in $h$. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 + c = 4x^2 + 4x + (1 + c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function $g$ so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

64. We need a function $g$ so that $g(f(x)) = g(x + 4) = h(x) = 4x - 4 = (x + 4) - 17$. So we see that the function $g$ must be $g(x) = 4x - 17$.

65. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} = h(x)$$

Because $h(-x) = h(x)$, $h$ is an even function.

66. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can’t simplify the expression, so we might try to find a counterexample to show that $h$ is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose $f$ is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so $h$ is odd if both $f$ and $g$ are odd.

Now suppose $f$ is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so $h$ is even if $g$ is odd and $f$ is even.
1. (a) \( y = y(t) = 10t - 4.9t^2 \). At \( t = 1.5 \), \( y = 10(1.5) - 4.9(1.5)^2 = 3.975 \). The average velocity between times 1.5 and 1.5 + h is

\[
\frac{v_{\text{ave}}}{h} = \frac{y(1.5+h) - y(1.5)}{(1.5+h) - 1.5} = \frac{[10(1.5+h) - 4.9(1.5+h)^2] - 3.975}{h} - \frac{15 + 10h - 11.025 - 14.7h - 4.9h^2 - 3.975}{h} = -4.7h - 4.9h^2 = -4.7 - 4.9h, \text{ if } h \neq 0.
\]

(i) [1.5, 2]: \( h = 0.5, v_{\text{ave}} = -7.15 \text{ m/s} \)

(ii) [1.5, 1.6]: \( h = 0.1, v_{\text{ave}} = -5.19 \text{ m/s} \)

(iii) [1.5, 1.55]: \( h = 0.05, v_{\text{ave}} = -4.945 \text{ m/s} \)

(iv) [1.5, 1.51]: \( h = 0.01, v_{\text{ave}} = -4.749 \text{ m/s} \)

(b) The instantaneous velocity when \( t = 1.5 \) (\( h \) approaches 0) is \(-4.7 \text{ m/s} \).

2. (a) The average velocity between \( t \) and \( t + h \) seconds is

\[
\frac{58(t + h) - 0.83(t + h)^2 - (58t - 0.83t^2)}{h} = \frac{58h - 1.66th - 0.83h^2}{h} = 58 - 1.66t - 0.83h \text{ if } h \neq 0.
\]

(a) Here \( t = 1 \), so the average velocity is \( 58 - 1.66 - 0.83h = 56.34 - 0.83h \).

(i) [1, 2]: \( h = 1.55 \) m/s

(ii) [1, 1.5]: \( h = 0.5, 55.925 \text{ m/s} \)

(iii) [1, 1.1]: \( h = 0.1, 56.257 \text{ m/s} \)

(iv) [1, 1.01]: \( h = 0.01, 56.3317 \text{ m/s} \)

(v) [1, 1.001]: \( h = 0.001, 56.33917 \text{ m/s} \)

(b) The instantaneous velocity after 1 second is 56.34 m/s.

3. (a) As \( x \) approaches 1, the values of \( f(x) \) approach 2, so

\[
\lim_{{x \to 1}} f(x) = 2.
\]

(b) As \( x \) approaches 3 from the left, the values of \( f(x) \) approach 1, so

\[
\lim_{{x \to 3^-}} f(x) = 1.
\]

(c) As \( x \) approaches 3 from the right, the values of \( f(x) \) approach 4, so

\[
\lim_{{x \to 3^+}} f(x) = 4.
\]

(d) \( \lim_{{x \to 2}} f(x) \) does not exist since the left-hand limit does not equal the right-hand limit.

(e) When \( x = 3, y = 3 \), so \( f(3) = 3 \).

4. (a) \( \lim_{{x \to 0}} f(x) = 3 \)

(b) \( \lim_{{x \to 3^-}} f(x) = 4 \)

(c) \( \lim_{{x \to 3^+}} f(x) = 2 \)

(d) \( \lim_{{x \to 3^+}} f(x) \) does not exist because the limits in part (b) and part (c) are not equal.

(e) \( f(3) = 3 \)
5. (a) \( \lim_{t \to 0^-} g(t) = -1 \)  
(b) \( \lim_{t \to 0^+} g(t) = -2 \)

(c) \( \lim_{t \to 0} g(t) \) does not exist because the limits in part (a) and part (b) are not equal.

(d) \( \lim_{t \to 2^-} g(t) = 2 \)  
(e) \( \lim_{t \to 2^+} g(t) = 0 \)

(f) \( \lim_{t \to 2} g(t) \) does not exist because the limits in part (d) and part (e) are not equal.

(g) \( g(2) = 1 \)  
(h) \( \lim_{t \to 4} g(t) = 3 \)

6. From the graph of

\[
 f(x) = \begin{cases} 
 1 + \sin x & \text{if } x < 0 \\
 \cos x & \text{if } 0 \leq x \leq \pi, \\
 \sin x & \text{if } x > \pi 
\end{cases}
\]

we see that \( \lim_{x \to a} f(x) \) exists for all \( a \) except \( a = \pi \). Notice that the right and left limits are different at \( a = \pi \).

7. \( \lim_{x \to 0^-} f(x) = -1 \), \( \lim_{x \to 0^+} f(x) = 2 \), \( f(0) = 1 \)

8. \( \lim_{x \to 0^-} f(x) = 1 \), \( \lim_{x \to 0^+} f(x) = -1 \), \( \lim_{x \to -2^-} f(x) = 0 \), \( \lim_{x \to -2^+} f(x) = 1 \), \( f(2) = 1 \), \( f(0) \) is undefined
9. \( \lim_{x \to 3^+} f(x) = 4, \lim_{x \to 3^-} f(x) = 2, \lim_{x \to 2} f(x) = 2, \)
\[ f(3) = 3, \quad f(-2) = 1 \]

10. \( \lim_{x \to 0^-} f(x) = 2, \lim_{x \to 0^+} f(x) = 0, \lim_{x \to 4^-} f(x) = 3, \)
\[ \lim_{x \to 4^+} f(x) = 0, \quad f(0) = 2, \quad f(4) = 1 \]

11. For \( f(x) = \frac{x^2 - 2x}{x^2 - x - 2} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>0.714286</td>
<td>1.9</td>
<td>0.655172</td>
</tr>
<tr>
<td>2.1</td>
<td>0.677419</td>
<td>1.95</td>
<td>0.661017</td>
</tr>
<tr>
<td>2.05</td>
<td>0.672131</td>
<td>1.99</td>
<td>0.665552</td>
</tr>
<tr>
<td>2.01</td>
<td>0.667774</td>
<td>1.995</td>
<td>0.666110</td>
</tr>
<tr>
<td>2.005</td>
<td>0.667221</td>
<td>1.999</td>
<td>0.666556</td>
</tr>
<tr>
<td>2.001</td>
<td>0.666778</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 3} \frac{x^2 - 2x}{x^2 - x - 2} = 0.8 = \frac{2}{3} \).
12. For \( f(x) = \frac{x^2 - 2x}{x^3 - x - 2} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>-1</td>
</tr>
<tr>
<td>-0.9</td>
<td>-9</td>
</tr>
<tr>
<td>-0.95</td>
<td>-19</td>
</tr>
<tr>
<td>-0.99</td>
<td>-99</td>
</tr>
<tr>
<td>-0.999</td>
<td>-999</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1.5</td>
<td>3</td>
</tr>
<tr>
<td>-1.1</td>
<td>11</td>
</tr>
<tr>
<td>-1.01</td>
<td>101</td>
</tr>
<tr>
<td>-1.001</td>
<td>1001</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to -1} \frac{x^2 - 2x}{x^3 - x - 2} \) does not exist since

\( f(x) \to \infty \) as \( x \to -1^- \) and \( f(x) \to -\infty \) as \( x \to -1^+ \).

13. For \( f(x) = \frac{\sin x}{x + \tan x} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>±1</td>
<td>0.329033</td>
</tr>
<tr>
<td>±0.5</td>
<td>0.458209</td>
</tr>
<tr>
<td>±0.2</td>
<td>0.493331</td>
</tr>
<tr>
<td>±0.1</td>
<td>0.498333</td>
</tr>
<tr>
<td>±0.05</td>
<td>0.499583</td>
</tr>
<tr>
<td>±0.01</td>
<td>0.499983</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 0} \frac{\sin x}{x + \tan x} = 0.5 = \frac{1}{2} \).

14. For \( f(h) = \frac{(2 + h)^5 - 32}{h} \):

<table>
<thead>
<tr>
<th>( h )</th>
<th>( f(h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>131.312500</td>
</tr>
<tr>
<td>0.1</td>
<td>88.410100</td>
</tr>
<tr>
<td>0.01</td>
<td>80.804010</td>
</tr>
<tr>
<td>0.001</td>
<td>80.080040</td>
</tr>
<tr>
<td>0.0001</td>
<td>80.008000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( h )</th>
<th>( f(h) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>48.812500</td>
</tr>
<tr>
<td>-0.1</td>
<td>72.390100</td>
</tr>
<tr>
<td>-0.01</td>
<td>79.203990</td>
</tr>
<tr>
<td>-0.001</td>
<td>79.920040</td>
</tr>
<tr>
<td>-0.0001</td>
<td>79.992000</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{h \to 0} \frac{(2 + h)^5 - 32}{h} = 80 \).
15. For \( f(x) = \frac{\sqrt{x+4} - 2}{x} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.236068</td>
</tr>
<tr>
<td>0.5</td>
<td>0.242641</td>
</tr>
<tr>
<td>0.1</td>
<td>0.248457</td>
</tr>
<tr>
<td>0.05</td>
<td>0.249224</td>
</tr>
<tr>
<td>0.01</td>
<td>0.249844</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.267949</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.258343</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.251582</td>
</tr>
<tr>
<td>-0.05</td>
<td>0.250786</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.250156</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4} \).

16. For \( f(x) = \frac{\tan 3x}{\tan 5x} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm 0.2 )</td>
<td>0.439279</td>
</tr>
<tr>
<td>( \pm 0.1 )</td>
<td>0.566236</td>
</tr>
<tr>
<td>( \pm 0.05 )</td>
<td>0.591893</td>
</tr>
<tr>
<td>( \pm 0.01 )</td>
<td>0.599680</td>
</tr>
<tr>
<td>( \pm 0.001 )</td>
<td>0.599997</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 0} \frac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5} \).

17. For \( f(x) = \frac{x^6 - 1}{x^{10} - 1} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.985337</td>
</tr>
<tr>
<td>0.9</td>
<td>0.719397</td>
</tr>
<tr>
<td>0.95</td>
<td>0.660186</td>
</tr>
<tr>
<td>0.99</td>
<td>0.612018</td>
</tr>
<tr>
<td>0.999</td>
<td>0.601200</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.183369</td>
</tr>
<tr>
<td>1.1</td>
<td>0.484119</td>
</tr>
<tr>
<td>1.05</td>
<td>0.540783</td>
</tr>
<tr>
<td>1.01</td>
<td>0.588022</td>
</tr>
<tr>
<td>1.001</td>
<td>0.598800</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 1} \frac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5} \).
18. For \( f(x) = \frac{9^x - 5^x}{x} \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.527864</td>
</tr>
<tr>
<td>0.1</td>
<td>0.711120</td>
</tr>
<tr>
<td>0.05</td>
<td>0.646496</td>
</tr>
<tr>
<td>0.01</td>
<td>0.599082</td>
</tr>
<tr>
<td>0.001</td>
<td>0.588906</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>−0.5</td>
<td>0.227761</td>
</tr>
<tr>
<td>−0.1</td>
<td>0.485984</td>
</tr>
<tr>
<td>−0.05</td>
<td>0.534447</td>
</tr>
<tr>
<td>−0.01</td>
<td>0.576706</td>
</tr>
<tr>
<td>−0.001</td>
<td>0.586869</td>
</tr>
</tbody>
</table>

It appears that \( \lim_{x \to 0} \frac{9^x - 5^x}{x} = 0.59 \). Later we will be able to show that the exact value is \( \ln(9/5) \).

19. (a) From the graphs, it seems that \( \lim_{x \to 0} \frac{\cos 2x - \cos x}{x^2} = -1.5 \).

20. (a) From the graphs, it seems that \( \lim_{x \to 0} \frac{\sin x}{\sin \pi x} = 0.32 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm 0.1 )</td>
<td>−1.493759</td>
</tr>
<tr>
<td>( \pm 0.01 )</td>
<td>−1.499938</td>
</tr>
<tr>
<td>( \pm 0.001 )</td>
<td>−1.499999</td>
</tr>
<tr>
<td>( \pm 0.0001 )</td>
<td>−1.500000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm 0.1 )</td>
<td>0.323068</td>
</tr>
<tr>
<td>( \pm 0.01 )</td>
<td>0.318357</td>
</tr>
<tr>
<td>( \pm 0.001 )</td>
<td>0.318310</td>
</tr>
<tr>
<td>( \pm 0.0001 )</td>
<td>0.318310</td>
</tr>
</tbody>
</table>

Later we will be able to show that the exact value is \( \frac{1}{\pi} \).
21. For $f(x) = x^2 - (2^x/1000)$:

(a)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.998000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.638259</td>
</tr>
<tr>
<td>0.6</td>
<td>0.358484</td>
</tr>
<tr>
<td>0.4</td>
<td>0.158680</td>
</tr>
<tr>
<td>0.2</td>
<td>0.038851</td>
</tr>
<tr>
<td>0.1</td>
<td>0.008928</td>
</tr>
<tr>
<td>0.05</td>
<td>0.001465</td>
</tr>
</tbody>
</table>

It appears that $\lim_{x \to 0} f(x) = 0$.

(b)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.000572</td>
</tr>
<tr>
<td>0.02</td>
<td>-0.000614</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.000907</td>
</tr>
<tr>
<td>0.005</td>
<td>-0.000978</td>
</tr>
<tr>
<td>0.003</td>
<td>-0.000993</td>
</tr>
<tr>
<td>0.001</td>
<td>-0.001000</td>
</tr>
</tbody>
</table>

It appears that $\lim_{x \to 0} f(x) = -0.001$. 
22. For \( h(x) = \frac{\tan x - x}{x^2} \).

(a)\[
\begin{array}{|c|c|}
\hline
x & h(x) \\
\hline
1.0 & 0.55740773 \\
0.5 & 0.37041992 \\
0.1 & 0.33467209 \\
0.05 & 0.33366700 \\
0.01 & 0.33334667 \\
0.005 & 0.33333667 \\
\hline
\end{array}
\]

(b) It seems that \( \lim_{x \to 0} h(x) = \frac{1}{2} \).

(c)\[
\begin{array}{|c|c|}
\hline
x & h(x) \\
\hline
0.001 & 0.33333350 \\
0.0005 & 0.33333344 \\
0.0001 & 0.33333000 \\
0.00005 & 0.33333600 \\
0.00001 & 0.33300000 \\
0.000001 & 0.00000000 \\
\hline
\end{array}
\]

Here the values will vary from one
calculator to another. Every calculator will
eventually give false values.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

23. The leftmost question mark is the solution of \( \sqrt{\pi} = 1.6 \) and the rightmost, \( \sqrt{\pi} = 2.4 \). So the values are \( 1.6^2 = 2.56 \) and
\( 2.4^2 = 5.76 \). On the left side, we need \( |x - 4| < |2.56 - 4| = 1.44 \). On the right side, we need \( |x - 4| < |5.76 - 4| = 1.76 \).

To satisfy both conditions, we need the more restrictive condition to hold—namely, \( |x - 4| < 1.44 \). Thus, we can choose
\( \delta = 1.44 \), or any smaller positive number.
24. The left-hand question mark is the positive solution of \( x^2 = \frac{1}{4} \), that is, \( x = \frac{1}{\sqrt{2}} \), and the right-hand question mark is the positive solution of \( x^2 = \frac{3}{4} \), that is, \( x = \sqrt{\frac{3}{2}} \). On the left side, we need \( |x - 1| < \left| 1 - \frac{1}{\sqrt{2}} \right| \approx 0.292 \) (rounding down to be safe). On the right side, we need \( |x - 1| < \left| \sqrt{\frac{3}{2}} - 1 \right| \approx 0.224 \). The more restrictive of these two conditions must apply, so we choose \( \delta = 0.224 \) (or any smaller positive number).

25. From the graph, we find that \( y = \tan x = 0.8 \) when \( x \approx 0.675 \), so \( \frac{\pi}{4} - \delta_1 \approx 0.675 \Rightarrow \delta_1 \approx \frac{\pi}{4} - 0.675 \approx 0.1106 \). Also, \( y = \tan x = 1.2 \) when \( x \approx 0.876 \), so \( \frac{\pi}{4} + \delta_2 \approx 0.876 \Rightarrow \delta_2 = 0.876 - \frac{\pi}{4} \approx 0.0906 \). Thus, we choose \( \delta = 0.0906 \) (or any smaller positive number) since this is the smaller of \( \delta_1 \) and \( \delta_2 \).

26. From the graph, we find that \( y = 2x/(x^2 + 4) = 0.3 \) when \( x = \frac{2}{3} \), so \( 1 - \delta_1 = \frac{2}{3} \Rightarrow \delta_1 = \frac{1}{3} \). Also, \( y = 2x/(x^2 + 4) = 0.5 \) when \( x = 2 \), so \( 1 + \delta_2 = 2 \Rightarrow \delta_2 = 1 \). Thus, we choose \( \delta = \frac{1}{3} \) (or any smaller positive number) since this is the smaller of \( \delta_1 \) and \( \delta_2 \).

27. (a) \( A = \pi r^2 \) and \( A = 1000 \text{ cm}^2 \) \( \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow r = \sqrt{\frac{1000}{\pi}} \) \( (r > 0) \approx 17.8412 \text{ cm} \).

(b) \( |A - 1000| \leq 5 \Rightarrow -5 \leq \pi r^2 - 1000 \leq 5 \Rightarrow 1000 - 5 \leq \pi r^2 \leq 1000 + 5 \Rightarrow \sqrt{\frac{295}{\pi}} \leq r \leq \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \leq r \leq 17.8858 \). \( \sqrt{\frac{295}{\pi}} - \sqrt{\frac{295}{\pi}} \approx 0.04466 \) and \( \sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455 \). So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 \text{ cm}^2 of 1000.

(c) \( x \) is the radius, \( f(x) \) is the area, \( a \) is the target radius given in part (a), \( L \) is the target area (1000), \( \varepsilon \) is the tolerance in the area (5), and \( \delta \) is the tolerance in the radius given in part (b).

28. (a) \( T = 0.1w^2 + 2.155w + 20 \) and \( T = 200 \) \( \Rightarrow 0.1w^2 + 2.155w + 20 = 200 \Rightarrow \) [by the quadratic formula or from the graph] \( w \approx 33.0 \text{ watts} (w > 0) \)

(b) From the graph, \( 199 \leq T \leq 201 \Rightarrow 32.89 < w < 33.11 \).

(c) \( x \) is the input power, \( f(x) \) is the temperature, \( a \) is the target input power given in part (a), \( L \) is the target temperature (200), \( \varepsilon \) is the tolerance in the temperature (1), and \( \delta \) is the tolerance in the power input in watts indicated in part (b) (0.11 watts).
29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then
\[ |(1 + \frac{1}{2}x) - 2| < \varepsilon. \]
But $|1 + \frac{1}{2}x - 2| < \varepsilon \Leftrightarrow |\frac{1}{2}x - 1| < \varepsilon \Leftrightarrow \frac{1}{2} |x - 3| < \varepsilon \Leftrightarrow |x - 3| < 2\varepsilon$. So if we choose $\delta = 3\varepsilon$, then
\[ 0 < |x - 3| < \delta \Rightarrow |(1 + \frac{1}{2}x) - 2| < \varepsilon. \]
Thus, $\lim_{x \to 3} (1 + \frac{1}{2}x) = 2$ by the definition of a limit.

30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|\frac{1}{2}x + 3 - 2| < \varepsilon$. But $|\frac{1}{2}x + 3 - 2| < \varepsilon \Leftrightarrow \frac{1}{2} |x + 2| < \varepsilon \Leftrightarrow |x - (-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then $0 < |x - (-2)| < \delta \Rightarrow |\frac{1}{2}x + 3 - 2| < \varepsilon$. Thus, $\lim_{x \to -2} (\frac{1}{2}x + 3) = 2$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then
\[ |(1 - 4x) - 13| < \varepsilon. \]
But $|(1 - 4x) - 13| < \varepsilon \Leftrightarrow |4x - 12| < \varepsilon \Leftrightarrow |4| |x + 3| < \varepsilon \Leftrightarrow |x - (-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \Rightarrow |(1 - 4x) - 13| < \varepsilon$. Thus, $\lim_{x \to -3} (1 - 4x) = 13$ by the definition of a limit.

32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then
\[ |(3x + 5) - (-1)| < \varepsilon. \]
But $|(3x + 5) - (-1)| < \varepsilon \Leftrightarrow |3x + 6| < \varepsilon \Leftrightarrow |3| |x + 2| < \varepsilon \Leftrightarrow |x + 2| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x + 2| < \delta \Rightarrow |(3x + 5) - (-1)| < \varepsilon$. Thus, $\lim_{x \to -2} (3x + 5) = -1$ by the definition of a limit.
33. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 1| < \delta \), then \( \left| \frac{2 + 4x}{3} - 2 \right| < \varepsilon \). But \( \left| \frac{2 + 4x}{3} - 2 \right| < \varepsilon \) \( \iff \) \( \left| \frac{4x - 4}{3} \right| < \varepsilon \) \( \iff \) \( \left| \frac{2}{3} \right| |x - 1| < \varepsilon \) \( \iff \) \( |x - 1| < \frac{3}{2} \varepsilon \). So if we choose \( \delta = \frac{3}{2} \varepsilon \), then \( 0 < |x - 1| < \delta \) \( \Rightarrow \) \( 2 + 4x - 2 < \frac{3}{2} \varepsilon \). Thus, \( \lim_{x \to 1} \frac{2 + 4x}{3} = 2 \) by the definition of a limit.

34. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 6| < \delta \), then \( \left| \frac{\frac{9}{4} + 3} - \frac{9}{2} \right| < \varepsilon \). \( \iff \) \( \left| \frac{9}{4} - \frac{9}{2} \right| < \varepsilon \) \( \iff \) \( \frac{1}{4} |x - 6| < \varepsilon \) \( \iff \) \( |x - 6| < 4 \varepsilon \). So choose \( \delta = 4 \varepsilon \). Then \( 0 < |x - 6| < \delta \) \( \Rightarrow \) \( |x - 6| < 4 \varepsilon \) \( \Rightarrow \) \( \frac{|x - 6|}{4} < \varepsilon \) \( \Rightarrow \) \( \frac{|x - 6|}{4} < \varepsilon \). By the definition of a limit, \( \lim_{x \to 6} \frac{\frac{9}{4} + 3}{x} = \frac{9}{2} \).

35. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then \( \left| \frac{x^2 + x - 6}{x - 2} - 5 \right| < \varepsilon \) \( \iff \) \( \left| \frac{(x + 3)(x - 2)}{x - 2} - 5 \right| < \varepsilon \) \( \iff \) \( |x + 3 - 5| < \varepsilon \) \( \iff \) \( |x - 2| < \varepsilon \). So choose \( \delta = \varepsilon \).

Then \( 0 < |x - 2| < \delta \) \( \Rightarrow \) \( |x - 2| < \varepsilon \) \( \Rightarrow \) \( |x + 3 - 5| < \varepsilon \) \( \Rightarrow \) \( \left| \frac{(x + 3)(x - 2)}{x - 2} - 5 \right| < \varepsilon \) \( \iff \) \( \left| \frac{x^2 + x - 6}{x - 2} - 5 \right| < \varepsilon \) \( \iff \) \( \left| \frac{x^2 + x - 6}{x - 2} - 5 \right| < \varepsilon \). By the definition of a limit, \( \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = 5 \).

36. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x + 1.5| < \delta \), then \( \left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \varepsilon \) \( \iff \) \( \left| \frac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6 \right| < \varepsilon \) \( \iff \) \( \left| 3 - 2x - 6 \right| < \varepsilon \) \( \iff \) \( |x - 1.5| < \varepsilon \) \( \iff \) \( 0 < |x + 1.5| < 2 \delta \) \( \Rightarrow \) \( |x + 1.5| < \varepsilon/2 \) \( \Rightarrow \) \( |x + 1.5| < \varepsilon \) \( \Rightarrow \) \( |x - 1.5| < \varepsilon \) \( \Rightarrow \) \( |x - 1.5| < \varepsilon/2 \) \( \Rightarrow \) \( |x - 1.5| < \varepsilon \) \( \Rightarrow \) \( |x - 1.5| < \varepsilon \) \( \Rightarrow \) \( \left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \varepsilon \) \( \iff \) \( \left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \varepsilon \).

By the definition of a limit, \( \lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6 \).

37. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \), then \( |x - a| < \varepsilon \). So \( \delta = \varepsilon \) will work.

38. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \), then \( |c - c| < \varepsilon \). But \( |c - c| = 0 \), so this will be true no matter what \( \delta \) we pick.

39. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \), then \( |x^2 - 0| < \varepsilon \) \( \iff \) \( x^2 < \varepsilon \) \( \iff \) \( |x| < \sqrt{\varepsilon} \). Take \( \delta = \sqrt{\varepsilon} \).

Then \( 0 < |x - 0| < \delta \) \( \Rightarrow \) \( |x^2 - 0| < \varepsilon \) \( \iff \) \( |x^2 - 0| < \varepsilon \). Thus, \( \lim_{x \to 0} x^2 = 0 \) by the definition of a limit.
40. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \), then \( |x^2 - 0| < \varepsilon \) \( \Rightarrow \) \( |x| < \sqrt[3]{\varepsilon} \). Take \( \delta = \sqrt[3]{\varepsilon} \).

Then \( 0 < |x - 0| < \delta \) \( \Rightarrow \) \( |x^2 - 0| < \delta^3 = \varepsilon \). Thus, \( \lim_{x \to 0} x^2 = 0 \) by the definition of a limit.

41. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \), then \( |x| < \varepsilon \). But \( |x| = |x| \). So this is true if we pick \( \delta = \varepsilon \). Thus, \( \lim_{x \to 0} |x| = 0 \) by the definition of a limit.

42. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 9 - \delta < x < 9 \), then \( \sqrt[3]{9 - x - 0} < \varepsilon \) \( \Leftrightarrow \) \( \sqrt[3]{9 - x} < \varepsilon \) \( \Leftrightarrow \) \( 9 - x < \varepsilon^4 \) \( \Leftrightarrow \) \( 9 - \varepsilon^4 < x < 9 \). So take \( \delta = \varepsilon^4 \). Then \( 9 - \delta < x < 9 \) \( \Leftrightarrow \) \( \sqrt[3]{9 - x - 0} < \varepsilon \). Thus, \( \lim_{x \to 9} \sqrt[3]{9 - x} = 0 \) by the definition of a limit.

43. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 3| < \delta \), then \( |x^2 - 9| < \varepsilon \) \( \Leftrightarrow \) \( |(x - 3)(x + 3)| < \varepsilon \). Notice that if \( |x - 3| < 1 \), then \(-1 < x - 3 < 1 \) \( \Rightarrow \) \( 5 < x + 3 < 7 \) \( \Rightarrow \) \( |x + 3| < 7 \). So take \( \delta = \min \{ 1, \varepsilon/7 \} \). Then \( 0 < |x - 3| < \delta \) \( \Leftrightarrow \) \( |(x - 3)(x + 3)| < |7(x - 3)| = 7 \cdot |x - 3| < 7\delta \leq \varepsilon \). Thus, \( \lim_{x \to 3} x^2 = 9 \) by the definition of a limit.

44. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 3| < \delta \), then \( |(x^2 + x - 4) - 8| < \varepsilon \) \( \Leftrightarrow \) \( |x^2 + x - 12| < \varepsilon \) \( \Leftrightarrow \) \( |(x - 3)(x + 4)| < \varepsilon \). Notice that if \( |x - 3| < 1 \), then \(-1 < x - 3 < 1 \) \( \Rightarrow \) \( 6 < x + 4 < 8 \) \( \Rightarrow \) \( |x + 4| < 8 \). So take \( \delta = \min \{ 1, \varepsilon/8 \} \). Then \( 0 < |x - 3| < \delta \) \( \Leftrightarrow \) \( |(x - 3)(x + 4)| \leq |8(x - 3)| = 8 \cdot |x - 3| < 8\delta \leq \varepsilon \). Thus, \( \lim_{x \to 3} (x^2 + x - 4) = 8 \) by the definition of a limit.

45. (a) The points of intersection in the graph are \((x_1, 2.6)\) and \((x_2, 3.4)\) with \(x_1 \approx 0.891\) and \(x_2 \approx 1.093\). Thus, we can take \( \delta \) to be the smaller of \( 1 - x_1 \) and \( x_2 - 1 \). So \( \delta = x_2 - 1 \approx 0.093 \).

(b) Solving \( x^2 + x + 1 = 3 + \varepsilon \) gives us two nonreal complex roots and one real root, which is
\[
x(\varepsilon) = \frac{(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2})^{1/3} - 12}{6(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2})^{1/3}}.
\]
Thus, \( \delta = x(\varepsilon) - 1 \).

(c) If \( \varepsilon = 0.4 \), then \( x(\varepsilon) \approx 1.093 \) and \( \delta = x(\varepsilon) - 1 \approx 0.093 \), which agrees with our answer in part (a).

46. Suppose that \( \lim_{t \to 0} H(t) = L \). Given \( \varepsilon = \frac{1}{2} \), there exists \( \delta > 0 \) such that if \( 0 < |t| < \delta \) \( \Rightarrow \) \( |H(t) - L| < \frac{1}{2} \) \( \Leftrightarrow \) \( L - \frac{1}{2} < H(t) < L + \frac{1}{2} \). For \( 0 < t < \delta \), \( H(t) = 1 \), so \( 1 < L + \frac{1}{2} \) \( \Rightarrow \) \( L > \frac{1}{2} \). For \( -\delta < t < 0 \), \( H(t) = 0 \), so \( L - \frac{1}{2} < 0 \) \( \Rightarrow \) \( L < \frac{1}{2} \). This contradicts \( L > \frac{1}{2} \). Therefore, \( \lim_{t \to 0} H(t) \) does not exist.
1.4-Part 1

1. (a) \[ \lim_{x \to 2} [f(x) + 5g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} [5g(x)] \]  
   \[ = \lim_{x \to 2} f(x) + 5 \lim_{x \to 2} g(x) \]  
   \[ = 4 + 5(-2) = -6 \]  
   [Limit Law 1]  
   \[ \lim_{x \to 2} g(x)^2 = \left( \lim_{x \to 2} g(x) \right)^2 \]  
   \[ = (-2)^2 = -8 \]  
   [Limit Law 6]  
   (c) \[ \lim_{x \to 2} \sqrt{f(x)} = \sqrt{\lim_{x \to 2} f(x)} \]  
   \[ = \sqrt{4} = 2 \]  
   [Limit Law 11]  
   (d) \[ \lim_{x \to 2} \frac{3f(x)}{g(x)} = \frac{\lim_{x \to 2} [3f(x)]}{\lim_{x \to 2} g(x)} \]  
   \[ = \frac{3 \lim_{x \to 2} f(x)}{\lim_{x \to 2} g(x)} \]  
   \[ = \frac{3(4)}{-2} = -6 \]  
   [Limit Law 5]  
   (e) Because the limit of the denominator is 0, we can’t use Limit Law 5. The given limit, \( \lim_{x \to 2} \frac{g(x)}{h(x)} \) does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.  
   (f) \[ \lim_{x \to 2} \frac{g(x) h(x)}{f(x)} = \frac{\lim_{x \to 2} [g(x) h(x)]}{\lim_{x \to 2} f(x)} \]  
   \[ = \frac{\lim_{x \to 2} g(x) \cdot \lim_{x \to 2} h(x)}{\lim_{x \to 2} f(x)} \]  
   \[ = \frac{-2 \cdot 0}{4} = 0 \]  
   [Limit Law 5]

2. (a) \[ \lim_{x \to 2} [f(x) + g(x)] = \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x) = 2 + 0 = 2 \]  
   (b) \( \lim_{x \to 1} g(x) \) does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.  
   (c) \[ \lim_{x \to 0} [f(x)g(x)] = \lim_{x \to 0} f(x) \cdot \lim_{x \to 0} g(x) = 0 \cdot 1.3 = 0 \]  
   (d) Since \( \lim_{x \to -1} g(x) = 0 \) and \( g \) is in the denominator, but \( \lim_{x \to -1} f(x) = -1 \neq 0 \), the given limit does not exist.  
   (e) \[ \lim_{x \to 2} x^2f(x) = \left[ \lim_{x \to 2} x^2 \right] \left[ \lim_{x \to 2} f(x) \right] = 2^2 \cdot 2 = 16 \]  
   (f) \[ \lim_{x \to 1} \sqrt{3 + f(x)} = \sqrt{3 + \lim_{x \to 1} f(x)} = \sqrt{3 + 1} = 2 \]

3. \[ \lim_{x \to 3} (5x^2 - 3x^2 + x - 6) = \lim_{x \to 3} (5x^2) - \lim_{x \to 3} (3x^2) + \lim_{x \to 3} x - \lim_{x \to 3} 6 \]  
   \[ = 5 \lim_{x \to 3} x^2 - 3 \lim_{x \to 3} x^2 + \lim_{x \to 3} x - \lim_{x \to 3} 6 \]  
   \[ = 5(3^2) - 3(3^2) + 3 - 6 \]  
   \[ = 105 \]  
   [Limit Laws 2 and 1]  
   [3]  
   [9, 8, and 7]
4. \( \lim_{t \to 1} (t^2 + 1)^2(t + 3)^6 = \left( \lim_{t \to 1} (t^2 + 1) \right)^2 \cdot \left( \lim_{t \to 1} (t + 3) \right)^6 \) [Limit Law 4]  
\[ = \left[ \lim_{t \to 1} (t^2 + 1) \right]^2 \cdot \left[ \lim_{t \to 1} (t + 3) \right]^6 \]  
\[ = \left[ \lim_{t \to 1} t^2 + \lim_{t \to 1} 1 \right]^2 \cdot \left[ \lim_{t \to 1} t + \lim_{t \to 1} 3 \right]^6 \] [1]  
\[ = \left[ (-1)^2 + 1 \right]^3 \cdot \left[ -1 + 3 \right]^6 = 8 \cdot 32 = 256 \] [9, 7, and 8]  

5. \( \lim_{t \to -2} \frac{t^4 - 2}{2t^2 - 3t + 2} = \frac{\lim_{t \to -2} (t^4 - 2)}{\lim_{t \to -2} (2t^2 - 3t + 2)} \) [Limit Law 5]  
\[ = \frac{\lim_{t \to -2} t^4 - \lim_{t \to -2} 2}{\lim_{t \to -2} 2t^2 - 3t + \lim_{t \to -2} 2} \] [1, 2, and 3]  
\[ = \frac{16 - 2}{2(4) - 3(-2) + 2} \] [9, 7, and 8]  
\[ = \frac{14}{16} = \frac{7}{8} \]  

6. \( \lim_{u \to -2} \sqrt{u^4 + 3u + 6} = \sqrt{\lim_{u \to -2} (u^4 + 3u + 6)} \) [11]  
\[ = \sqrt{\lim_{u \to -2} u^4 + \lim_{u \to -2} 3u + \lim_{u \to -2} 6} \] [1, 2, and 3]  
\[ = \sqrt{(-2)^4 + 3(-2) + 6} \] [9, 8, and 7]  
\[ = \sqrt{16 - 6 + 6} = \sqrt{16} = 4 \]  

7. \( \lim_{x \to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3) = \lim_{x \to 8} (1 + \sqrt[3]{x}) \cdot \lim_{x \to 8} (2 - 6x^2 + x^3) \) [Limit Law 4]  
\[ = \left( \lim_{x \to 8} 1 + \lim_{x \to 8} \sqrt[3]{x} \right) \cdot \left( \lim_{x \to 8} 2 - \lim_{x \to 8} 6x^2 + \lim_{x \to 8} x^3 \right) \] [1, 2, and 3]  
\[ = (1 + \sqrt[3]{8}) \cdot (2 - 6 \cdot 8^2 + 8^3) \] [7, 10, 9]  
\[ = (1 + 2) \cdot (2 - 6 \cdot 64 + 512) \]  
\[ = 3(130) = 390 \]  

8. \( \lim_{x \to 0} \frac{\cos x}{5 + 2x^2} = \frac{\lim_{x \to 0} \cos x}{\lim_{x \to 0} (5 + 2x^2)} \) [5]  
\[ = \frac{\left( \lim_{x \to 0} \cos x \right)^4}{\lim_{x \to 0} 5 + 2 \lim_{x \to 0} x^2} \] [6, 1, and 3]  
\[ = \frac{1^4}{5 + 2(0)^2} = \frac{1}{5} \] [7, 9, and Equation 1]
9. \[ \lim_{\theta \to \pi/2} \theta \sin \theta = \left( \lim_{\theta \to \pi/2} \theta \right) \left( \lim_{\theta \to \pi/2} \sin \theta \right) \]
   \[ = \frac{\pi}{2} \cdot \sin \left( \frac{\pi}{2} \right) \]
   \[ = \frac{\pi}{2} \]

10. (a) The left-hand side of the equation is not defined for \( x = 2 \), but the right-hand side is.

    (b) Since the equation holds for all \( x \neq 2 \), it follows that both sides of the equation approach the same limit as \( x \to 2 \), just as in Example 3. Remember that in finding \( \lim_{x \to a} f(x) \), we never consider \( x = a \).

11. \[ \lim_{x \to 5} \frac{x^2 - 6x + 5}{x - 5} = \lim_{x \to 5} \frac{(x - 5)(x - 1)}{x - 5} = \lim_{x \to 5} (x - 1) = 5 - 1 = 4 \]

12. \[ \lim_{x \to -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \lim_{x \to -4} \frac{(x + 4)(x + 1)}{(x + 4)(x - 1)} = \lim_{x \to -4} \frac{x + 1}{x - 1} = \frac{-4 + 1}{-4 - 1} = \frac{-3}{-5} = \frac{3}{5} \]

13. \[ \lim_{x \to 5} \frac{x^2 - 5x + 6}{x - 5} \]
   does not exist since \( x - 5 \to 0 \), but \( x^2 - 5x + 6 \to 6 \) as \( x \to 5 \).

14. \[ \lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4} \]
   does not exist since \( x^2 - 3x - 4 \to 0 \) but \( x^2 - 4x \to 5 \) as \( x \to -1 \).

15. \[ \lim_{t \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3} = \lim_{t \to -3} \frac{(t + 3)(t - 3)}{(2t + 1)(t + 3)} = \lim_{t \to -3} \frac{t - 3}{2t + 1} = \frac{-3 - 3}{2(-3) + 1} = \frac{-6}{-5} = \frac{6}{5} \]

16. \[ \lim_{x \to -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3} = \lim_{x \to -1} \frac{(2x + 1)(x + 1)}{(x - 3)(x + 1)} = \lim_{x \to -1} \frac{2x + 1}{x - 3} = \frac{2(-1) + 1}{-1 - 3} = \frac{-1}{-4} = \frac{1}{4} \]

17. \[ \lim_{h \to 0} \frac{(-5 + h)^2 - 25}{h} = \lim_{h \to 0} \frac{(25 - 10h + h^2) - 25}{h} = \lim_{h \to 0} \frac{-10h + h^2}{h} = \lim_{h \to 0} \frac{h(-10 + h)}{h} = \lim_{h \to 0} (-10 + h) = -10 \]

18. \[ \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} \cdot \frac{\sqrt{1 + h} + 1}{\sqrt{1 + h} + 1} \]
   \[ = \lim_{h \to 0} \frac{(1 + h) - 1}{h \sqrt{1 + h} + 1} = \lim_{h \to 0} \frac{h}{h \sqrt{1 + h} + 1} = \frac{1}{2} \]

19. By the formula for the sum of cubes, we have
   \[ \lim_{x \to -2} \frac{x + 2}{x^2 + 8} = \lim_{x \to -2} \frac{x + 2}{(x + 2)(x^2 - 2x + 4)} = \lim_{x \to -2} \frac{1}{x^2 - 2x + 4} = \frac{1}{4 + 4 + 4} = \frac{1}{12} \]
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20. \[ \lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1} = \frac{\lim_{x \to -1} (x + 1)^2}{(x^2 + 1)(x^2 - 1)} = \frac{(x + 1)^2}{(x^2 + 1)(x + 1)(x - 1)} \]
   \[ = \lim_{x \to -1} \frac{x + 1}{(x^2 + 1)(x - 1)} = \frac{0}{2(-2)} = 0 \]

21. \[ \lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h} = \frac{\lim_{h \to 0} (\sqrt{9 + h} - 3) \cdot (\sqrt{9 + h} + 3)}{\lim_{h \to 0} h (\sqrt{9 + h} + 3)} = \frac{\lim_{h \to 0} (\sqrt{9 + h})^2 - 9^2}{\lim_{h \to 0} h (\sqrt{9 + h} + 3)} \]
   \[ = \lim_{h \to 0} \frac{\frac{h}{\sqrt{9 + h} + 3}}{\sqrt{9 + h} + 3} = \lim_{h \to 0} \frac{1}{3 + 3} = \frac{1}{6} \]

22. \[ \lim_{h \to 0} \frac{(3 + h)^{-1} - 3^{-1}}{h} = \lim_{h \to 0} \frac{\frac{1}{3 + h} - \frac{1}{3}}{h} = \lim_{h \to 0} \frac{3 - (3 + h)}{h(3 + h)3} = \lim_{h \to 0} \frac{-h}{h(3 + h)3} \]
   \[ = \lim_{h \to 0} \left[ -\frac{1}{3(3 + h)} \right] = \frac{-1}{3(3 + 0)} = -\frac{1}{9} \]

23. \[ \lim_{x \to 16} \frac{4 - \sqrt{x}}{16 - x} = \lim_{x \to 16} \frac{(4 - \sqrt{x})(4 + \sqrt{x})}{(16 - x)(4 + \sqrt{x})} = \lim_{x \to 16} \frac{16 - x}{x(16 - x)(4 + \sqrt{x})} \]
   \[ = \lim_{x \to 16} \frac{1}{x(4 + \sqrt{x})} = \frac{1}{16(4 + \sqrt{16})} = \frac{1}{16(8)} = \frac{1}{128} \]

24. \[ \lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + t} \right) = \lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t(t + 1)} \right) = \lim_{t \to 0} \frac{t + 1 - 1}{t + 1} = \lim_{t \to 0} \frac{1}{t + 1} = \frac{1}{0 + 1} = 1 \]

25. \[ \lim_{x \to -4} \frac{4 + x}{4 + x} = \lim_{x \to -4} \frac{x + 4}{4x} = \lim_{x \to -4} \frac{x + 4}{4x} = \lim_{x \to -4} \frac{1}{4(-4)} = -\frac{1}{16} \]

26. \[ \lim_{x \to -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4} = \lim_{x \to -4} \frac{(\sqrt{x^2 + 9} - 5)(\sqrt{x^2 + 9} + 5)}{(x + 4)(\sqrt{x^2 + 9} + 5)} \]
   \[ = \lim_{x \to -4} \frac{x^2 - 16}{(x + 4)(\sqrt{x^2 + 9} + 5)} = \lim_{x \to -4} \frac{(x + 4)(x - 4)}{(x + 4)(\sqrt{x^2 + 9} + 5)} \]
   \[ = \lim_{x \to -4} \frac{x - 4}{\sqrt{x^2 + 9} + 5} = \frac{-4 - 4}{\sqrt{16} + 9 + 5} = \frac{-8}{5 + 5} = -\frac{4}{5} \]

27. \[ \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{(x^2 + 3xh + 3xh^2 + h^2) - x^2}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^2}{h} \]
   \[ = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 \]
28. \[ \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{(x+h)^2} - \frac{1}{x^2} \right) = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h(x+h)^2} = \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{-h(2x+h)}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{-2x}{x^2} \cdot \frac{2}{x^3} = -\frac{2}{x^3} \]

29. (a) 

\[ \lim_{x \to 0} \frac{x}{\sqrt{1+3x} - 1} \approx \frac{2}{3} \]

(b) 

\[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
-0.001 & 0.6661663 \\
-0.0001 & 0.6666667 \\
-0.000001 & 0.6666667 \\
0.000001 & 0.6666667 \\
0.0001 & 0.6666667 \\
0.001 & 0.6671663 \\
\hline
\end{array}
\]

The limit appears to be \( \frac{2}{3} \).

(c) \[ \lim_{x \to 0} \left( \frac{x}{\sqrt{1+3x} - 1} \cdot \frac{\sqrt{1+3x} + 1}{\sqrt{1+3x} + 1} \right) = \lim_{x \to 0} \frac{x(\sqrt{1+3x} + 1)}{(1+3x) - 1} = \lim_{x \to 0} \frac{x(\sqrt{1+3x} + 1)}{3x} \]

\[ = \frac{1}{3} \lim_{x \to 0} (\sqrt{1+3x} + 1) \quad [\text{Limit Law 3}] \]

\[ = \frac{1}{3} \left( \sqrt{\lim_{x \to 0} (1+3x) + \lim_{x \to 0} 1} \right) \quad [1 \text{ and } 11] \]

\[ = \frac{1}{3} \left( \sqrt{1 + 3 \lim_{x \to 0} x + 1} \right) \quad [1, 3, \text{ and } 7] \]

\[ = \frac{1}{3} (\sqrt{1 + 3 \cdot 0 + 1}) \quad [7 \text{ and } 8] \]

\[ = \frac{1}{3} (1 + 1) = \frac{2}{3} \]
30. (a) \[
\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \approx 0.29
\]

(b) \[
\begin{array}{c|c}
 x & f(x) \\
\hline
 -0.001 & 0.2886992 \\
 -0.0001 & 0.2886775 \\
 -0.00001 & 0.2886754 \\
 -0.000001 & 0.2886752 \\
 0.000001 & 0.2886751 \\
 0.00001 & 0.2886749 \\
 0.001 & 0.2886727 \\
 0.01 & 0.2886511 \\
\end{array}
\]

The limit appears to be approximately 0.2887.

(c) \[
\lim_{x \to 0} \left( \frac{\sqrt{3+x} - \sqrt{3}}{x} \cdot \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} \right) = \lim_{x \to 0} \frac{(3+x) - 3}{x (\sqrt{3+x} + \sqrt{3})} = \lim_{x \to 0} \frac{1}{\sqrt{3+x} + \sqrt{3}}
\]

\[= \frac{1}{\sqrt{3} + \frac{1}{\sqrt{3}}} \quad \text{[Limit Laws 5 and 1]} \]

\[= \frac{1}{2 \sqrt{3}} \quad \text{[7 and 11]} \]

31. Let \( f(x) = -x^2, g(x) = x^2 \cos 20\pi x \) and \( h(x) = x^2 \). Then

\[-1 \leq \cos 20\pi x \leq 1 \implies -x^2 \leq x^2 \cos 20\pi x \leq x^2 \implies f(x) \leq g(x) \leq h(x). \]

So since \( \lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \), by the Squeeze Theorem we have

\[\lim_{x \to 0} g(x) = 0.\]

32. Let \( f(x) = -\sqrt{x^2 + x^2}, g(x) = \sqrt{x^2 + x^2} \sin(\pi/x), h(x) = \sqrt{x^2 + x^2} \). Then

\[-1 \leq \sin(\pi/x) \leq 1 \implies -\sqrt{x^2 + x^2} \leq \sqrt{x^2 + x^2} \sin(\pi/x) \leq \sqrt{x^2 + x^2} \implies f(x) \leq g(x) \leq h(x). \]

So since \( \lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \), by the Squeeze Theorem we have \( \lim_{x \to 0} g(x) = 0 \).

33. We have \( \lim_{x \to 4} (4x - 9) = 4(4) - 9 = 7 \) and \( \lim_{x \to 4} (x^2 - 4x + 7) = 4^2 - 4(4) + 7 = 7 \). Since \( 4x - 9 \leq f(x) \leq x^2 - 4x + 7 \) for \( x \geq 0 \), \( \lim_{x \to 4} f(x) = 7 \) by the Squeeze Theorem.
34. We have \( \lim_{x \to 1} (2x) = 2 \) and \( \lim_{x \to 1} (x^4 - x^2 + 2) = 1^4 - 1^2 + 2 = 2 \). Since \( 2x \leq g(x) \leq x^4 - x^2 + 2 \) for all \( x \), \( \lim_{x \to 1} g(x) = 2 \) by the Squeeze Theorem.

35. \(-1 \leq \cos(2/x) \leq 1 \) \( \Rightarrow \) \(-x^4 \leq x^4 \cos(2/x) \leq x^4 \). Since \( \lim_{x \to 0} (-x^4) = 0 \) and \( \lim_{x \to 0} x^4 = 0 \), we have \( \lim_{x \to 0} [x^4 \cos(2/x)] = 0 \) by the Squeeze Theorem.

36. \(-1 \leq \sin(2\pi/x) \leq 1 \) \( \Rightarrow \) \( 0 \leq \sin^2(2\pi/x) \leq 1 \) \( \Rightarrow \) \( 1 \leq 1 + \sin^2(2\pi/x) \leq 2 \) \( \Rightarrow \) \( x \leq \sqrt{x} [1 + \sin^2(2\pi/x)] \leq 2\sqrt{x} \). Since \( \lim_{x \to 0^+} \sqrt{x} = 0 \) and \( \lim_{x \to 0^+} 2\sqrt{x} = 0 \), we have \( \lim_{x \to 0^+} [\sqrt{x} (1 + \sin^2(2\pi/x))] = 0 \) by the Squeeze Theorem.

37. \( |x - 3| = \begin{cases} x - 3 & \text{if } x - 3 \geq 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases} \)

Thus, \( \lim_{x \to 3^+} (2x + |x - 3|) = \lim_{x \to 3^+} (2x + x - 3) = \lim_{x \to 3^+} (3x - 3) = 3(3) - 3 = 6 \) and

\( \lim_{x \to 3^-} (2x + |x - 3|) = \lim_{x \to 3^-} (2x + 3 - x) = \lim_{x \to 3^-} (x + 3) = 3 + 3 = 6 \). Since the left and right limits are equal, \( \lim_{x \to 3} (2x + |x - 3|) = 6 \).

38. \( |x + 6| = \begin{cases} x + 6 & \text{if } x + 6 \geq 0 \\ -(x + 6) & \text{if } x + 6 < 0 \end{cases} \)

We'll look at the one-sided limits.

\( \lim_{x \to -6^+} \frac{2x + 12}{|x + 6|} = \lim_{x \to -6^+} \frac{2(x + 6)}{x + 6} = 2 \) and \( \lim_{x \to -6^-} \frac{2x + 12}{|x + 6|} = \lim_{x \to -6^-} \frac{2(x + 6)}{-(x + 6)} = -2 \)

The left and right limits are different, so \( \lim_{x \to -6} \frac{2x + 12}{|x + 6|} \) does not exist.

39. \( |2x^2 - x^2| = |x^2(2x - 1)| = |x^2| \cdot |2x - 1| = x^2 |2x - 1| \)

\( |2x - 1| = \begin{cases} 2x - 1 & \text{if } 2x - 1 \geq 0 \\ -(2x - 1) & \text{if } 2x - 1 < 0 \end{cases} = \begin{cases} 2x - 1 & \text{if } x \geq 0.5 \\ -(2x - 1) & \text{if } x < 0.5 \end{cases} \)

So \( |2x^2 - x^2| = x^2 [-(2x - 1)] \) for \( x < 0.5 \).

Thus, \( \lim_{x \to 0.5^-} \frac{2x - 1}{|2x^2 - x^2|} = \lim_{x \to 0.5^-} \frac{2x - 1}{x^2 [-(2x - 1)]} = \lim_{x \to 0.5^-} \frac{-1}{(0.5)^2} = \frac{-1}{0.25} = -4 \).

40. Since \( |x| = -x \) for \( x < 0 \), we have \( \lim_{x \to -2} \frac{2 - |x|}{2 + x} = \lim_{x \to -2} \frac{2 - (-x)}{2 + x} = \lim_{x \to -2} \frac{2 + x}{2 + x} = \lim_{x \to -2} 1 = 1 \).
41. Since \(|x| = -x\) for \(x < 0\), we have \(\lim_{x \to 0^-} \left( \frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \to 0^-} \left( \frac{1}{x} - \frac{1}{-x} \right) = \lim_{x \to 0^-} \frac{2}{x}\), which does not exist since the denominator approaches 0 and the numerator does not.

42. Since \(|x| = x\) for \(x > 0\), we have \(\lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{x} \right) = \lim_{x \to 0^+} 0 = 0\).

43. (a) \(\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} \frac{x^2 + x - 6}{|x - 2|} = \lim_{x \to 2^+} \frac{(x+3)(x-2)}{|x-2|}\)

\[= \lim_{x \to 2^+} \frac{(x+3)(x-2)}{x-2} \quad \text{[since } x - 2 > 0 \text{ if } x \to 2^+]\]

\[= \lim_{x \to 2^+} (x+3) = 5\]

(ii) The solution is similar to the solution in part (i), but now \(|x - 2| = 2 - x\) since \(x - 2 < 0\) if \(x \to 2^-\).

Thus, \(\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} -(x + 3) = -5\).

(b) Since the right-hand and left-hand limits of \(g\) at \(x = 2\) are not equal, \(\lim_{x \to 2} g(x)\) does not exist.
44. (a) If \( x \to 1^+ \), then \( x > 1 \) and \( g(x) = x - 1 \). Thus, \( \lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (x - 1) = 1 - 1 = 0 \).

(b) If \( x \to 1^- \), then \( x < 1 \) and \( g(x) = 1 - x^2 \). Thus, \( \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (1 - x^2) = 1 - 1^2 = 0 \).

Since the left- and right-hand limits of \( g \) at 1 are equal, \( \lim_{x \to 1} g(x) = 0 \).

(iii) If \( x \to 0 \), then \( -1 < x < 1 \) and \( g(x) = 1 - x^2 \). Thus, \( \lim_{x \to 0} g(x) = \lim_{x \to 0} (1 - x^2) = 1 - 0^2 = 1 \).

(iv) If \( x \to -1^- \), then \( x < -1 \) and \( g(x) = -x \). Thus, \( \lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} (-x) = -(-1) = 1 \).

(v) If \( x \to -1^+ \), then \( -1 < x < 1 \) and \( g(x) = 1 - x^2 \). Thus,

\[
\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (1 - x^2) = 1 - (-1)^2 = 1 - 1 = 0
\]

(vi) \( \lim_{x \to -1} g(x) \) does not exist because the limits in part (iv) and part (v) are not equal.

45. (a) \( \lfloor x \rfloor = -2 \) for \(-2 \leq x < -1\), so \( \lim_{x \to -2^+} \lfloor x \rfloor = \lim_{x \to -2^+} (-2) = -2 \)

(ii) \( \lfloor x \rfloor = -3 \) for \(-3 \leq x < -2\), so \( \lim_{x \to -2^-} \lfloor x \rfloor = \lim_{x \to -2^-} (-3) = -3 \).

The right and left limits are different, so \( \lim_{x \to -2} \lfloor x \rfloor \) does not exist.

(iii) \( \lfloor x \rfloor = -3 \) for \(-3 \leq x < -2\), so \( \lim_{x \to -2^+} \lfloor x \rfloor = \lim_{x \to -2^-} (-3) = -3 \).

(b) \( \lfloor x \rfloor = n - 1 \) for \( n - 1 \leq x < n \), so \( \lim_{x \to n^-} \lfloor x \rfloor = \lim_{x \to n^-} (n - 1) = n - 1 \).

(ii) \( \lfloor x \rfloor = n \) for \( n \leq x < n + 1 \), so \( \lim_{x \to n^+} \lfloor x \rfloor = \lim_{x \to n^+} n = n \).

(c) \( \lim_{x \to \alpha} \lfloor x \rfloor \) exists \( \iff \alpha \) is not an integer.
46. (a) 

(\text{Diagram of a graph})

(b) 
(i) \( \lim_{{x \to n^-}} f(x) = \lim_{{x \to n^-}} (x - [x]) = \lim_{{x \to n^-}} [x - (n - 1)] \\
= n - (n - 1) = 1 \\

(ii) \( \lim_{{x \to n^+}} f(x) = \lim_{{x \to n^+}} (x - [x]) = \lim_{{x \to n^+}} (x - n) = n - n = 0 \\

(c) \( \lim_{{x \to \alpha}} f(x) \) exists \( \iff \) \( \alpha \) is not an integer.

47. The graph of \( f(x) = [x] + [-x] \) is the same as the graph of \( g(x) = -1 \) with holes at each integer, since \( f(\alpha) = 0 \) for any integer \( \alpha \). Thus, \( \lim_{{x \to -2^-}} f(x) = -1 \) and \( \lim_{{x \to -2^+}} f(x) = -1 \), so \( \lim_{{x \to -2}} f(x) = -1 \). However, \( f(2) = [2] + [-2] = 2 + (-2) = 0 \), so \( \lim_{{x \to 2}} f(x) \neq f(2) \).

48. \( \lim_{{v \to c^+}} \left( \frac{L_o \sqrt{1 - \frac{v^2}{c^2}}}{v} \right) = L_o \sqrt{1 - 1} = 0 \). As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since \( L \) is not defined for \( v > c \).

49. \( \lim_{{x \to 0}} \frac{\sin 3x}{x} = \lim_{{x \to 0}} \frac{3 \sin 3x}{3x} \) [multiply numerator and denominator by 3] 

\[ = 3 \lim_{{x \to 0}} \frac{\sin 3x}{3x} \] [as \( x \to 0, 3x \to 0 \)]

\[ = 3 \lim_{{\theta \to 0}} \frac{\sin \theta}{\theta} \] [let \( \theta = 3x \)]

\[ = 3(1) \] [Equation 2]

\[ = 3 \]

50. \( \lim_{{x \to 0}} \frac{\sin 4x}{\sin 6x} = \lim_{{x \to 0}} \left( \frac{\sin 4x}{x} \cdot \frac{x}{\sin 6x} \right) = \lim_{{x \to 0}} \frac{4 \sin 4x}{4x} \cdot \lim_{{x \to 0}} \frac{6x}{6 \sin 6x} \)

\[ = 4 \lim_{{x \to 0}} \frac{\sin 4x}{4x} \cdot \frac{1}{6} \lim_{{x \to 0}} \frac{6x}{\sin 6x} = 4(1) \cdot \frac{1}{6}(1) = \frac{2}{3} \]
51. \( \lim_{t \to 0} \frac{\tan 6t}{\sin 2t} = \lim_{t \to 0} \left( \frac{\sin 6t}{t} \cdot \frac{1}{\cos 6t} \cdot \frac{t}{\sin 2t} \right) = \lim_{t \to 0} \frac{\sin 6t}{6t} \cdot \lim_{t \to 0} \frac{1}{\cos 6t} \cdot \lim_{t \to 0} \frac{2t}{2 \sin 2t} = 6(1) \cdot \frac{1}{1} \cdot \frac{1}{2} = 3 \)

52. \( \lim_{t \to 0} \frac{\sin^2 3t}{t^2} = \lim_{t \to 0} \left( \frac{\sin 3t}{t} \cdot \frac{\sin 3t}{t} \right) = \lim_{t \to 0} \frac{\sin 3t}{t} \cdot \lim_{t \to 0} \frac{\sin 3t}{t} = \left( \lim_{t \to 0} \frac{\sin 3t}{t} \right)^2 = (3 \cdot 1)^2 = 9 \)

53. \( \lim_{x \to 0} \frac{\sin 3x}{5x^2 - 4x} = \lim_{x \to 0} \left( \frac{\sin 3x}{3x} \cdot \frac{3}{5x^2 - 4x} \right) = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \lim_{x \to 0} \frac{3}{5x^2 - 4x} = 1 \cdot \left( \frac{3}{-4} \right) = -\frac{3}{4} \)

54. \( \lim_{x \to 0} \frac{\sin 3x \sin 5x}{x^2} = \lim_{x \to 0} \left( \frac{\sin 3x}{3x} \cdot \frac{5 \sin 5x}{5x} \right) = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \lim_{x \to 0} \frac{5 \sin 5x}{5x} = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 5 \lim_{x \to 0} \frac{\sin 5x}{5x} = 3(1 \cdot 1) = 15 \)

55. Divide numerator and denominator by \( \theta \). (\( \sin \theta \) also works.)

\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta + \tan \theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \frac{1}{1 + \frac{\tan \theta}{\cos \theta}} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta} = \frac{1}{1 + 1} = \frac{1}{2} \]

56. \( \lim_{x \to 0} \frac{\sin(x^2)}{x} = \lim_{x \to 0} \left[ \frac{x \cdot \sin(x^2)}{x \cdot x} \right] = \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{\sin(x^2)}{x^2} = 0 \cdot \lim_{y \to 0^+} \frac{\sin y}{y} \quad [\text{where } y = x^2] = 0 \cdot 1 = 0 \)

57. Since \( p(x) \) is a polynomial, \( p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \). Thus, by the Limit Laws,

\[ \lim_{x \to a} p(x) = \lim_{x \to a} \left( a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \right) = a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \cdots + a_n \lim_{x \to a} x^n = a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n = p(a) \]

Thus, for any polynomial \( p \), \( \lim_{x \to a} p(x) = p(a) \).

58. Let \( r(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are any polynomials, and suppose that \( q(a) \neq 0 \). Then

\[ \lim_{x \to a} r(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)} \quad [\text{Limit Law 5}] = \frac{p(a)}{q(a)} \quad [\text{Exercise 57}] = r(a). \]

59. \( \lim_{h \to 0} \sin(a + h) = \lim_{h \to 0} (\sin a \cos h + \cos a \sin h) = \lim_{h \to 0} (\sin a \cos h) + \lim_{h \to 0} (\cos a \sin h) \)

\[ = \left( \lim_{h \to 0} \sin a \right) \left( \lim_{h \to 0} \cos h \right) + \left( \lim_{h \to 0} \cos a \right) \left( \lim_{h \to 0} \sin h \right) = (\sin a)(1) + (\cos a)(0) = \sin a \]
60. As in the previous exercise, we must show that \( \lim_{h \to 0} \cos(a + h) = \cos a \) to prove that the cosine function has the Direct Substitution Property.

\[
\lim_{h \to 0} \cos(a + h) = \lim_{h \to 0} (\cos a \cos h - \sin a \sin h) = \lim_{h \to 0} (\cos a \cos h) - \lim_{h \to 0} (\sin a \sin h)
\]
\[
= (\lim_{h \to 0} \cos a) (\lim_{h \to 0} \cos h) - (\lim_{h \to 0} \sin a) (\lim_{h \to 0} \sin h) = (\cos a)(1) - (\sin a)(0) = \cos a
\]

61. \( \lim_{x \to 1} |f(x) - 8| = \lim_{x \to 1} \left[ \frac{f(x) - 8}{x - 1} \cdot (x - 1) \right] = \lim_{x \to 1} \frac{f(x) - 8}{x - 1} \cdot \lim_{x \to 1} (x - 1) = 10 \cdot 0 = 0. \)

Thus, \( \lim_{x \to 1} f(x) = \lim_{x \to 1} |f(x) - 8| + \lim_{x \to 1} 8 = 0 + 8 = 8. \)

Note: The value of \( \lim_{x \to 1} \frac{f(x) - 8}{x - 1} \) does not affect the answer since it's multiplied by 0. What's important is that \( \lim_{x \to 1} f(x) - 8 \) exists.

62. We use an indirect proof. Suppose that \( \lim_{x \to a} \frac{f(x)}{g(x)} = L \) exists. Then by Property 5

\[
\lim_{x \to a} f(x) = \lim_{x \to a} \left[ \frac{f(x)}{g(x)} \cdot g(x) \right] = \lim_{x \to a} \frac{f(x)}{g(x)} \cdot \lim_{x \to a} g(x) = L \cdot 0 = 0
\]

But this contradicts the assumption that \( \lim_{x \to a} f(x) \neq 0 \). Therefore \( \lim_{x \to a} \frac{f(x)}{g(x)} \) does not exist.

63. Let \( f(x) = [x] \) and \( g(x) = -[x] \). Then \( \lim_{x \to \frac{1}{2}} f(x) \) and \( \lim_{x \to \frac{1}{2}} g(x) \) do not exist \hspace{1cm} \text{[Example 8]} \]

but \( \lim_{x \to \frac{1}{2}} [f(x) + g(x)] = \lim_{x \to \frac{1}{2}} ([x] - [x]) = \lim_{x \to \frac{1}{2}} 0 = 0. \)

64. Let \( f(x) = H(x) \) and \( g(x) = 1 - H(x) \), where \( H \) is the Heaviside function defined in Exercise 1.2.59.

Thus, either \( f \) or \( g \) is 0 for any value of \( x \). Then \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) do not exist, but \( \lim_{x \to 0} [f(x)g(x)] = \lim_{x \to 0} 0 = 0. \)

65. Since the denominator approaches 0 as \( x \to -2 \), the limit will exist only if the numerator also approaches 0 as \( x \to -2 \). In order for this to happen, we need \( \lim_{x \to -2} (3x^2 + ax + a + 3) = 0 \) \hspace{1cm} \Leftrightarrow \hspace{1cm} 3(-2)^2 + a(-2) + a + 3 = 0 \Leftrightarrow 12 - 2a + a + 3 = 0 \Leftrightarrow a = 15. \) With \( a = 15 \), the limit becomes

\[
\lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \to -2} \frac{3(x + 2)(x + 3)}{(x - 1)(x + 2)} = \lim_{x \to -2} \frac{3(x + 3)}{x - 1} = \lim_{x \to -2} \frac{3(-2 + 3)}{-2 - 1} = \frac{3}{-3} = -1.
\]
66. **Solution 1:** First, we find the coordinates of \(P\) and \(Q\) as functions of \(r\). Then we can find the equation of the line determined by these two points, and thus find the \(x\)-intercept (the point \(R\)), and take the limit as \(r \to 0\). The coordinates of \(P\) are \((0, r)\).

The point \(Q\) is the point of intersection of the two circles \(x^2 + y^2 = r^2\) and \((x - 1)^2 + y^2 = 1\). Eliminating \(y\) from these equations, we get \(r^2 - x^2 = 1 - (x - 1)^2 \iff r^2 = 1 + 2x - 1 \iff x = \frac{1}{2} r^2\). Substituting back into the equation of the shrinking circle to find the \(y\)-coordinate, we get \((\frac{1}{2} r^2)^2 + y^2 = r^2 \iff y^2 = r^2 (1 - \frac{1}{4} r^2) \iff y = r \sqrt{1 - \frac{1}{4} r^2}\) (the positive \(y\)-value). So the coordinates of \(Q\) are \((\frac{1}{2} r^2, r \sqrt{1 - \frac{1}{4} r^2})\). The equation of the line joining \(P\) and \(Q\) is thus \(y - r = \frac{r \sqrt{1 - \frac{1}{4} r^2} - r}{\frac{1}{2} r^2 - 0} (x - 0)\). We set \(y = 0\) in order to find the \(x\)-intercept, and get

\[
x = \frac{-r \frac{\frac{1}{2} r^2}{r \left(\sqrt{1 - \frac{1}{4} r^2} - 1\right)}}{\frac{-1}{2} r^2 \left(\frac{\sqrt{1 - \frac{1}{4} r^2} + 1}{1 - \frac{1}{4} r^2 - 1}\right)} = 2 \left(\sqrt{1 - \frac{1}{4} r^2} + 1\right)
\]

Now we take the limit as \(r \to 0^+\): \(\lim_{r \to 0^+} x = \lim_{r \to 0^+} 2 \left(\sqrt{1 - \frac{1}{4} r^2} + 1\right) = \lim_{r \to 0^+} 2 \left(\sqrt{1} + 1\right) = 4\).

So the limiting position of \(R\) is the point \((4, 0)\).

**Solution 2:** We add a few lines to the diagram, as shown. Note that \(\angle PQS = 90^\circ\) (subtended by diameter \(PS\)). So \(\angle SQR = 90^\circ = \angle QOT\) (subtended by diameter \(OT\)). It follows that \(\angle QOS = \angle TQR\). Also \(\angle PSQ = 90^\circ - \angle SPQ = \angle ORP\). Since \(\triangle QOS\) is isosceles, so is \(\triangle QTR\), implying that \(QT = TR\). As the circle \(C_2\) shrinks, the point \(Q\) plainly approaches the origin, so the point \(R\) must approach a point twice as far from the origin as \(T\), that is, the point \((4, 0)\), as above.
1. From Definition 1, \( \lim_{x \to 4} f(x) = f(4) \).

2. The graph of \( f \) has no hole, jump, or vertical asymptote.

3. (a) \( f \) is discontinuous at \(-4\) since \( f(-4) \) is not defined and at \(-2\), \( 2 \), and \( 4 \) since the limit does not exist (the left and right limits are not the same).

   (b) \( f \) is continuous from the left at \(-2\) since \( \lim_{x \to -2^-} f(x) = f(-2) \). \( f \) is continuous from the right at \( 2 \) and \( 4 \) since \( \lim_{x \to 2^+} f(x) = f(2) \) and \( \lim_{x \to 4^+} f(x) = f(4) \). It is continuous from neither side at \(-4\) since \( f(-4) \) is undefined.

4. \( g \) is continuous on \([-4, -2), (-2, 2), [2, 4), (4, 6), \) and \((6, 8)\).

5. The graph of \( y = f(x) \) must have a discontinuity at \( x = 2 \) and must show that \( \lim_{x \to 2^+} f(x) = f(2) \).

6. The graph of \( y = f(x) \) must have discontinuities at \( x = -1 \) and \( x = 4 \). It must show that \( \lim_{x \to -1^-} f(x) = f(-1) \) and \( \lim_{x \to 4^+} f(x) = f(4) \).
7. The graph of \( y = f(x) \) must have a removable discontinuity (a hole) at \( x = 3 \) and a jump discontinuity at \( x = 5 \).

8. The graph of \( y = f(x) \) must have a discontinuity at \( x = -2 \) with \( \lim_{x \to -2^-} f(x) \neq f(-2) \) and \( \lim_{x \to -2^+} f(x) \neq f(-2) \). It must also show that \( \lim_{x \to -2^-} f(x) = f(2) \) and \( \lim_{x \to -2^+} f(x) \neq f(2) \).

9. (a) \( \text{Cost (in dollars)} \)

(b) There are discontinuities at times \( t = 1, 2, 3, \) and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.
10. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from Paris increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from Paris increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

11. If \( f \) and \( g \) are continuous and \( g(2) = 6 \), then \( \lim_{x \to 2} [3f(x) + f(x) \cdot g(x)] = 36 \Rightarrow 3 \lim_{x \to 2} f(x) + \lim_{x \to 2} f(x) \cdot \lim_{x \to 2} g(x) = 36 \Rightarrow 3f(2) + f(2) \cdot 6 = 36 \Rightarrow 9f(2) = 36 \Rightarrow f(2) = 4. \)

12. \( \lim_{x \to 2} f(x) = \lim_{x \to 2} \left(3x^4 - 5x + \sqrt[3]{x^2 + 4}\right) = 3 \lim_{x \to 2} x^4 - 5 \lim_{x \to 2} x + \lim_{x \to 2} \sqrt[3]{x^2 + 4} \)

\[ = 3(2)^4 - 5(2) + \sqrt[3]{2^2 + 4} = 48 - 10 + 2 = 40 = f(2). \]

By the definition of continuity, \( f \) is continuous at \( a = 2 \).

13. \( \lim_{x \to -1} f(x) = \lim_{x \to -1} (x + 2x^3)^4 = \left( \lim_{x \to -1} x + 2 \lim_{x \to -1} x^3 \right)^4 = [-1 + 2(-1)^3]^4 = (-3)^4 = 81 = f(-1). \)

By the definition of continuity, \( f \) is continuous at \( a = -1 \).

14. For \(-4 < a < 4\) we have \( \lim_{z \to a} f(x) = \lim_{z \to a} x \sqrt{16 - x^2} = \lim_{z \to a} x \sqrt{16} - \lim_{z \to a} x^2 = a \sqrt{16 - a^2} = f(a) \), so \( f \) is continuous on \((-4, 4)\). Similarly, we get \( \lim_{z \to a} f(x) = 0 = f(4) \) and \( \lim_{z \to 4} f(x) = 0 = f(-4) \), so \( f \) is continuous from the left at 4 and from the right at -4. Thus, \( f \) is continuous on \([-4, 4]\).

15. \( f(x) = \frac{1}{x^2 + 2} \) is discontinuous at \( a = -2 \) because \( f(-2) \) is undefined.
16. \( f(x) = \begin{cases} 
\frac{1}{x-1} & \text{if } x \neq 1 \\
2 & \text{if } x = 1 
\end{cases} \) is discontinuous at 1 because \\
\( \lim_{x \to 1} f(x) \) does not exist.

17. \( f(x) = \begin{cases} 
1 - x^2 & \text{if } x < 1 \\
1/x & \text{if } x \geq 1 
\end{cases} \)

The left-hand limit of \( f \) at \( a = 1 \) is \\
\( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - x^2) = 0. \) The right-hand limit of \( f \) at \( a = 1 \) is \\
\( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1/x) = 1. \) Since these limits are not equal, \( \lim_{x \to 1} f(x) \) \\
does not exist and \( f \) is discontinuous at 1.

18. \( f(x) = \begin{cases} 
\frac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\
1 & \text{if } x = 1 
\end{cases} \)

\( \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2} \), \\
but \( f(1) = 1 \), so \( f \) is discontinuous at 1.

19. \( F(x) = \frac{2x^2 - x - 1}{x^2 + 1} \) is a rational function, so it is continuous on its domain, \((-\infty, \infty)\), by Theorem 5(b). 

20. By Theorem 6, the root function \( \sqrt{x} \) and the polynomial function \( 1 + x^2 \) are continuous on \( \mathbb{R} \). By part 4 of Theorem 4, the 
    product \( G(x) = \sqrt{x} (1 + x^2) \) is continuous on its domain, \( \mathbb{R} \).

21. \( x^2 - 2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2} \), so \( Q(x) = \frac{\sqrt{x^2 - 2}}{x^2 - 2} \) has domain \((-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)\). 

Now \( x^2 - 2 \) is 
    continuous everywhere by Theorem 5(a) and \( \sqrt{x - 2} \) is continuous everywhere by Theorems 5(a), 7, and 9. Thus, \( Q \) is 
    continuous on its domain by part 5 of Theorem 4.

22. By Theorem 6, the trigonometric function \( \sin x \) and the polynomial function \( x + 1 \) are continuous on \( \mathbb{R} \). By part 5 of 
Theorem 4, \( h(x) = \frac{\sin x}{x + 1} \) is continuous on its domain, \( \{x \mid x \neq -1\} \).
23. \( M(x) = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} \) is defined when \( \frac{x+1}{x} \geq 0 \) \( \Rightarrow \) \( x+1 \geq 0 \) and \( x > 0 \) or \( x+1 \leq 0 \) and \( x < 0 \) \( \Rightarrow \) \( x > 0 \) or \( x \leq -1 \), so \( M \) has domain \((-\infty, -1] \cup (0, \infty)\). \( M \) is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

24. The sine and cosine functions are continuous everywhere by Theorem 6, so \( F(x) = \sin(\cos(\sin x)) \), which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 8.

25. \( y = \frac{1}{1+\sin x} \) is undefined and hence discontinuous when \( 1 + \sin x = 0 \) \( \iff \) \( \sin x = -1 \) \( \iff \) \( x = -\frac{\pi}{2} + 2\pi n \), \( n \) an integer. The figure shows discontinuities for \( n = -1, 0, \) and \( 1 \); that is, \( -\frac{5\pi}{2} \approx -7.85, -\frac{\pi}{2} \approx -1.57, \) and \( \frac{3\pi}{2} \approx 4.71 \).

26. The function \( y = f(x) = \tan \sqrt{x} \) is continuous throughout its domain because it is the composite of a trigonometric function and a root function. The square root function has domain \([0, \infty)\) and the tangent function has domain \( \{x \mid x \neq \frac{\pi}{2} + \pi n\} \). So \( f \) is discontinuous when \( x < 0 \) and when \( \sqrt{x} = \frac{\pi}{2} + \pi n \) \( \Rightarrow \) \( x = \left( \frac{\pi}{2} + \pi n \right)^2 \), where \( n \) is a nonnegative integer. Note that as \( x \) increases, the distance between discontinuities increases.

27. Because we are dealing with root functions, \( 5 + \sqrt{x} \) is continuous on \([0, \infty)\), \( \sqrt{x} + 5 \) is continuous on \([-5, \infty)\), so the quotient \( f(x) = \frac{5 + \sqrt{x}}{\sqrt{5 + x}} \) is continuous on \([0, \infty)\). Since \( f \) is continuous at \( x = 4 \), \( \lim_{x \to 4} f(x) = f(4) = \frac{7}{3} \).

28. Because \( x \) is continuous on \( \mathbb{R} \), \( \sin x \) is continuous on \( \mathbb{R} \), and \( x + \sin x \) is continuous on \( \mathbb{R} \), the composite function \( f(x) = \sin(x + \sin x) \) is continuous on \( \mathbb{R} \), so \( \lim_{x \to \pi} f(x) = f(\pi) = \sin(\pi + \sin \pi) = \sin \pi = 0 \).
29. \( f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases} \)

By Theorem 5, since \( f(x) \) equals the polynomial \( x^2 \) on \((-\infty, 1)\), \( f \) is continuous on \((-\infty, 1)\). By Theorem 7, since \( f(x) \) equals the root function \( \sqrt{x} \) on \((1, \infty)\), \( f \) is continuous on \((1, \infty)\). At \( x = 1 \), \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} x^2 = 1 \) and \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} \sqrt{x} = 1 \). Thus, \( \lim_{x \to 1} f(x) \) exists and equals 1. Also, \( f(1) = \sqrt{1} = 1 \). Thus, \( f \) is continuous at \( x = 1 \).

We conclude that \( f \) is continuous on \((-\infty, \infty)\).

30. \( f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases} \)

By Theorem 7, the trigonometric functions are continuous. Since \( f(x) = \sin x \) on \((-\infty, \pi/4)\) and \( f(x) = \cos x \) on \((\pi/4, \infty)\), \( f \) is continuous on \((-\infty, \pi/4) \cup (\pi/4, \infty)\). \( \lim_{x \to (\pi/4)^-} f(x) = \lim_{x \to (\pi/4)^-} \sin x = \sin \frac{\pi}{4} = 1/\sqrt{2} \) since the sine function is continuous at \( \pi/4 \). Similarly, \( \lim_{x \to (\pi/4)^+} f(x) = \lim_{x \to (\pi/4)^+} \cos x = 1/\sqrt{2} \) by continuity of the cosine function at \( \pi/4 \). Thus, \( \lim_{x \to (\pi/4)} f(x) \) exists and equals \( 1/\sqrt{2} \), which agrees with the value \( f(\pi/4) \). Therefore, \( f \) is continuous at \( \pi/4 \), so \( f \) is continuous on \((-\infty, \infty)\).

31. \( f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases} \)

\( f \) is continuous on \((-\infty, 0)\), \((0, 1)\), and \((1, \infty)\) since on each of these intervals it is a polynomial. Now \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} (x + 2) = 2 \) and \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} 2x^2 = 0 \), so \( f \) is discontinuous at 0. Since \( f(0) = 0 \), \( f \) is continuous from the right at 0. Also \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} 2x^2 = 2 \) and \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (2 - x) = 1 \), so \( f \) is discontinuous at 1. Since \( f(1) = 2 \), \( f \) is continuous from the left at 1.

32. By Theorem 5, each piece of \( F \) is continuous on its domain. We need to check for continuity at \( r = R \).

\[
\lim_{r \to R^-} F(r) = \lim_{r \to R^-} \frac{GMr}{R^3} = \frac{GM}{R^2} \quad \text{and} \quad \lim_{r \to R^+} F(r) = \lim_{r \to R^+} \frac{GM}{r^2} = \frac{GM}{R^2},
\]

so \( \lim_{r \to R} F(r) = \frac{GM}{R^2} \). Since \( F(R) = \frac{GM}{R^2} \), \( F \) is continuous at \( R \). Therefore, \( F \) is a continuous function of \( r \).
33. \( f(x) = \begin{cases} \frac{cx^2 + 2x}{x^2 - cx} & \text{if } x < 2 \\ \frac{x^3 - cx}{x^2 - 2x} & \text{if } x \geq 2 \end{cases} \)

\( f \) is continuous on \((\infty, 2)\) and \((2, \infty)\). Now \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{cx^2 + 2x}{x^2 - cx}\right) = 4c + 4 \) and

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{x^3 - cx}{x^2 - 2x}\right) = 8 - 2c. \]

So \( f \) is continuous \( \iff 4c + 4 = 8 - 2c \iff 6c = 4 \iff c = \frac{2}{3}. \)

Thus, for \( f \) to be continuous on \((\infty, \infty)\), \( c = \frac{2}{3}. \)

34. \( f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases} \)

At \( x = 2 \):

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} \left(\frac{(x+2)(x-2)}{x-2}\right) = \lim_{x \to 2^-} (x+2) = 2 + 2 = 4 \]

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3 \]

We must have \( 4a - 2b + 3 = 4 \), or \( 4a - 2b = 1 \) (1).

At \( x = 3 \):

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3 \]

\[ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x - a + b) = 6 - a + b \]

We must have \( 9a - 3b + 3 = 6 - a + b \), or \( 10a - 4b = 3 \) (2).

Now solve the system of equations by adding \(-2\) times equation (1) to equation (2):

\[ -8a + 4b = -2 \]
\[ 10a - 4b = 3 \]

\[ \frac{2a}{2} = 1 \]

So \( a = \frac{1}{2} \). Substituting \( \frac{1}{2} \) for \( a \) in (1) gives us \(-2b = -1\), so \( b = \frac{1}{2} \) as well. Thus, for \( f \) to be continuous on \((\infty, \infty)\),

\[ a = b = \frac{1}{2}. \]

35. (a) \( f(x) = \frac{x^4 - 1}{x - 1} = \frac{(x^2 + 1)(x^2 - 1)}{x - 1} = \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = (x^2 + 1)(x + 1) \) [or \( x^2 + x = x + 1 \)]

for \( x \neq 1 \). The discontinuity is removable and \( g(x) = x^2 + x + 1 \) agrees with \( f \) for \( x \neq 1 \) and is continuous on \( \mathbb{R} \).

(b) \( f(x) = \frac{x^2 - x^2 - 2x}{x - 2} = \frac{x(x - 2)(x + 1)}{x - 2} = x(x + 1) \) [or \( x^2 + x \)] for \( x \neq 2 \). The discontinuity is removable and \( g(x) = x^2 + x \) agrees with \( f \) for \( x \neq 2 \) and is continuous on \( \mathbb{R} \).

(c) \( \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} [\sin x] = 0 \) and \( \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} [\sin x] = \lim_{x \to \pi^+} (-1) = -1 \), so \( \lim_{x \to \pi} f(x) \) does not exist. The discontinuity at \( x = \pi \) is a jump discontinuity.
36. \[ f(x) = x^2 + 10 \sin x \] does not satisfy the conclusion of the Intermediate Value Theorem.

37. \( f(x) = x^2 + 10 \sin x \) is continuous on the interval \([31, 32]\), \( f(31) \approx 957 \), and \( f(32) \approx 1030 \). Since \( 957 < 1000 < 1030 \), there is a number \( c \) in \((31, 32)\) such that \( f(c) = 1000 \) by the Intermediate Value Theorem. \textit{Note:} There is also a number \( c \) in \((-32, -31)\) such that \( f(c) = 1000 \).

38. Suppose that \( f(3) < 6 \). By the Intermediate Value Theorem applied to the continuous function \( f \) on the closed interval \([2, 3]\), the fact that \( f(2) = 8 > 6 \) and \( f(3) < 6 \) implies that there is a number \( c \) in \((2, 3)\) such that \( f(c) = 6 \). This contradicts the fact that the only solutions of the equation \( f(x) = 6 \) are \( x = 1 \) and \( x = 4 \). Hence, our supposition that \( f(3) < 6 \) was incorrect. It follows that \( f(3) \geq 6 \). But \( f(3) \neq 6 \) because the only solutions of \( f(x) = 6 \) are \( x = 1 \) and \( x = 4 \). Therefore, \( f(3) > 6 \).

39. \( f(x) = x^4 + x - 3 \) is continuous on the interval \([1, 2]\), \( f(1) = -1 \), and \( f(2) = 15 \). Since \(-1 < 0 < 15\), there is a number \( c \) in \((1, 2)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( x^4 + x - 3 = 0 \) in the interval \((1, 2)\).

40. \( f(x) = \sqrt[3]{x} + x - 1 \) is continuous on the interval \([0, 1]\), \( f(0) = -1 \), and \( f(1) = 1 \). Since \(-1 < 0 < 1\), there is a number \( c \) in \((0, 1)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( \sqrt[3]{x} + x - 1 = 0 \), or \( \sqrt[3]{x} = 1 - x \), in the interval \((0, 1)\).

41. \( f(x) = \cos x - x \) is continuous on the interval \([0, 1]\), \( f(0) = 1 \), and \( f(1) = \cos 1 - 1 \approx -0.46 \). Since \(-0.46 < 0 < 1\), there is a number \( c \) in \((0, 1)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( \cos x - x = 0 \), or \( \cos x = x \), in the interval \((0, 1)\).

42. \( f(x) = \tan x - 2x \) is continuous on the interval \([0, 1.4]\), \( f(1) = \tan 1 - 2 \approx -0.44 \), and \( f(1.4) = \tan 1.4 - 2.8 \approx 3.00 \). Since \(-0.44 < 0 < 3.00\), there is a number \( c \) in \((0, 1.4)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( \tan x - 2x = 0 \), or \( \tan x = 2x \), in the interval \((0, 1.4)\).
43. (a) $f(x) = \cos x - x^2$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number $c$ in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^2 = 0$, or $\cos x = x^2$, in the interval $(0, 1)$.

(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

44. (a) $f(x) = x^5 - x^2 + 2x + 3$ is continuous on $[-1, 0]$, $f(-1) = -1 < 0$, and $f(0) = 3 > 0$. Since $-1 < 0 < 3$, there is a number $c$ in $(-1, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^2 + 2x + 3 = 0$ in the interval $(-1, 0)$.

(b) $f(-0.88) \approx -0.062 < 0$ and $f(-0.87) \approx 0.0047 > 0$, so there is a root between $-0.88$ and $-0.87$.

45. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number $c$ in $(1, 2)$ such that $f(c) = c^5 - c^2 - 4 = 0$.

(b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.

46. (a) Let $f(x) = \sqrt{x - 5} - \frac{1}{x + 3}$. Then $f(5) = -\frac{1}{6} < 0$ and $f(6) = \frac{5}{9} > 0$, and $f$ is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number $c$ in $(5, 6)$ such that $f(c) = 0$. This implies that $\frac{1}{c + 3} = \sqrt{c - 5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.

47. If there is such a number, it satisfies the equation $x^2 + 1 = x \Rightarrow x^2 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number $c$ between $-2$ and $-1$ such that $f(c) = 0$, so that $c = c^2 + 1$. 
1.5-Part 1

48. \[ \frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0 \implies a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0. \] Let \( p(x) \) denote the left side of the last equation. Since \( p \) is continuous on \([-1, 1]\), \( p(-1) = -4a < 0 \), and \( p(1) = 2b > 0 \), there exists a \( c \) in \((-1, 1)\) such that \( p(c) = 0 \) by the Intermediate Value Theorem. Note that the only root of either denominator that is in \((-1, 1)\) is 
\( (-1 + \sqrt{5})/2 = r \), but \( p(r) = (3\sqrt{5} - 9)a/2 \neq 0 \). Thus, \( c \) is not a root of either denominator, so \( p(c) = 0 \implies x = c \) is a root of the given equation.

49. \( f(x) = x^4 \sin(1/x) \) is continuous on \((\infty, 0) \cup (0, \infty)\) since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since \(-1 \leq \sin(1/x) \leq 1\), we have \(-x^4 \leq x^4 \sin(1/x) \leq x^4\). Because \( \lim_{x \to 0} (-x^4) = 0 \) and \( \lim_{x \to 0} x^4 = 0 \), the Squeeze Theorem gives us \( \lim_{x \to 0} (x^4 \sin(1/x)) = 0 \), which equals \( f(0) \). Thus, \( f \) is continuous at 0 and, hence, on \((\infty, \infty)\).
50. (a) \( \lim_{x \to 0^+} F(x) = 0 \) and \( \lim_{x \to 0^-} F(x) = 0 \), so \( \lim_{x \to 0} F(x) = 0 \), which is \( F(0) \), and hence \( F \) is continuous at \( x = a \) if \( a = 0 \). For \( a > 0 \), \( \lim_{x \to a} F(x) = \lim_{x \to a} x = a = F(a) \). For \( a < 0 \), \( \lim_{x \to a} F(x) = \lim_{x \to a} (-x) = -a = F(a) \). Thus, \( F \) is continuous at \( x = a \); that is, continuous everywhere.

(b) Assume that \( f \) is continuous on the interval \( I \). Then for \( a \in I \), \( \lim_{x \to a} |f(x)| = | \lim_{x \to a} f(x) | = |f(a)| \) by Theorem 8. (If \( a \) is an endpoint of \( I \), use the appropriate one-sided limit.) So \( |f| \) is continuous on \( I \).

(c) No, the converse is false. For example, the function \( f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \) is not continuous at \( x = 0 \), but \( |f(x)| = 1 \) is continuous on \( \mathbb{R} \).

51. Define \( u(t) \) to be the monk’s distance from the monastery, as a function of time \( t \) (in hours), on the first day, and define \( d(t) \) to be his distance from the monastery, as a function of time, on the second day. Let \( D \) be the distance from the monastery to the top of the mountain. From the given information we know that \( u(0) = 0 \), \( u(12) = D \), \( d(0) = D \) and \( d(12) = 0 \). Now consider the function \( u - d \), which is clearly continuous. We calculate that \( (u - d)(0) = -D \) and \( (u - d)(12) = D \).

So by the Intermediate Value Theorem, there must be some time \( t_0 \) between 0 and 12 such that \( (u - d)(t_0) = 0 \) \( \iff \) \( u(t_0) = d(t_0) \). So at time \( t_0 \) after 7:00 AM, the monk will be at the same place on both days.
1.6-Part 1

1. (a) \( \lim_{x \to -\infty} f(x) = -2 \)  
   (b) \( \lim_{x \to -\infty} f(x) = 2 \)  
   (c) \( \lim_{x \to 1} f(x) = \infty \)  
   (d) \( \lim_{x \to 2} f(x) = -\infty \)  
   (e) Vertical: \( x = 1, x = 3 \); horizontal: \( y = -2, y = 2 \)

2. (a) \( \lim_{x \to -\infty} g(x) = 2 \)  
   (b) \( \lim_{x \to -\infty} g(x) = -1 \)  
   (c) \( \lim_{x \to 0} g(x) = -\infty \)  
   (d) \( \lim_{x \to 2^-} g(x) = -\infty \)  
   (e) \( \lim_{x \to 2^+} g(x) = \infty \)  
   (f) Vertical: \( x = 0, x = 2 \); horizontal: \( y = -1, y = 2 \)

3. \( \lim_{x \to 0} f(x) = -\infty \), \[ \lim_{x \to -\infty} f(x) = 5, \] \[ \lim_{x \to \infty} f(x) = -5 \]

4. \( \lim_{x \to 0^+} f(x) = \infty \), \( \lim_{x \to 0^-} f(x) = -\infty \), \[ \lim_{x \to \infty} f(x) = 1, \lim_{x \to -\infty} f(x) = 1 \]
5. \( \lim_{x \to -2} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = \infty, \)

\( \lim_{x \to 0^-} f(x) = 0, \quad \lim_{x \to 0^+} f(x) = \infty, \)

\( \lim_{x \to 0^-} f(x) = -\infty \)

![Graph of a function with a vertical asymptote at x = 0 and horizontal asymptotes at y = -\infty and y = \infty.]

6. \( \lim_{x \to \infty} f(x) = 3, \)

\( \lim_{x \to -2^-} f(x) = \infty, \)

\( \lim_{x \to -2^+} f(x) = -\infty, \)

\( f \) is odd

![Graph of a function with a horizontal asymptote at y = 3 and vertical asymptotes at x = -2 and x = 2.]

Page 2
7. \( f(0) = 3, \quad \lim_{x \to 0^-} f(x) = 4, \)
\[
\lim_{x \to 0^+} f(x) = 2,
\]
\[
\lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to -4^-} f(x) = -\infty,
\]
\[
\lim_{x \to -4^+} f(x) = \infty, \quad \lim_{x \to \infty} f(x) = 3
\]

8. \( \lim_{x \to -2} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = 2, \)
\[f(0) = 0, \quad f \text{ is even}\]

9. If \( f(x) = \frac{x^2}{2^x} \), then a calculator gives \( f(0) = 0, \ f(1) = 0.5, \ f(2) = 1, \ f(3) = 1.125, \ f(4) = 1, \ f(5) = 0.78125, \)
\[
f(6) = 0.5625, \ f(7) = 0.3828125, \ f(8) = 0.25, \ f(9) = 0.15625, \ f(10) = 0.09765625, \ f(20) \approx 0.00038147,
\]
\[
f(50) \approx 2.2204 \times 10^{-12}, \ f(100) \approx 7.8886 \times 10^{-27}.
\]

It appears that \( \lim_{x \to \infty} \left(\frac{x^2}{2^x}\right) = 0. \)
10. (a) \( f(x) = \frac{1}{x^2 - 1} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>-1.14</td>
</tr>
<tr>
<td>0.9</td>
<td>-3.69</td>
</tr>
<tr>
<td>0.99</td>
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</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>1.00001</td>
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</tr>
</tbody>
</table>

From these calculations, it seems that \( \lim_{x \to 1^-} f(x) = -\infty \) and \( \lim_{x \to 1^+} f(x) = \infty \).

(b) If \( x \) is slightly smaller than 1, then \( x^2 - 1 \) will be a negative number close to 0, and the reciprocal of \( x^2 - 1 \), that is, \( f(x) \), will be a negative number with large absolute value. So \( \lim_{x \to 1^-} f(x) = -\infty \).

If \( x \) is slightly larger than 1, then \( x^2 - 1 \) will be a small positive number, and its reciprocal, \( f(x) \), will be a large positive number. So \( \lim_{x \to 1^+} f(x) = \infty \).

(c) It appears from the graph of \( f \) that \( \lim_{x \to 1^-} f(x) = -\infty \) and \( \lim_{x \to 1^+} f(x) = \infty \).

11. Vertical: \( x \approx -1.62, x \approx 0.62, x = 1 \);
   Horizontal: \( y = 1 \)

12. (a) From a graph of \( f(x) = (1 - 2/x)^n \) in a window of \([0, 10,000]\) by \([0, 0.2]\), we estimate that \( \lim_{x \to \infty} f(x) = 0.14 \) (to two decimal places.)

(b) From the table, we estimate that \( \lim_{x \to \infty} f(x) = 0.1353 \) (to four decimal places.)

<table>
<thead>
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<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
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<td>10,000</td>
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</tr>
<tr>
<td>100,000</td>
<td>0.135333</td>
</tr>
<tr>
<td>1,000,000</td>
<td>0.135335</td>
</tr>
</tbody>
</table>

13. \( \lim_{x \to -3^+} \frac{x+2}{x+3} = -\infty \) since the numerator is negative and the denominator approaches 0 from the positive side as \( x \to -3^+ \).
14. \( \lim_{x \to -5^-} \frac{6}{x - 5} = -\infty \) since \((x - 5) \to 0^\text{as} \ x \to 5^-\) and \(\frac{6}{x - 5} < 0\) for \(x < 5\).

15. \( \lim_{x \to 1} \frac{2 - x}{(x - 1)^2} = \infty \) since the numerator is positive and the denominator approaches 0 through positive values as \(x \to 1\).

16. \( \lim_{x \to -\pi^-} \cot x = \lim_{x \to -\pi^-} \frac{\cos x}{\sin x} = -\infty \) since the numerator is negative and the denominator approaches 0 through positive values as \(x \to -\pi^-\).

17. \( \lim_{x \to 2\pi^-} \csc x = \lim_{x \to 2\pi^-} \frac{x}{\sin x} = -\infty \) since the numerator is positive and the denominator approaches 0 through negative values as \(x \to 2\pi^-\).

18. \( \lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 4x + 4} = \lim_{x \to 2^-} \frac{x(x-2)}{(x-2)^2} = \lim_{x \to 2^-} \frac{x}{x-2} = -\infty \) since the numerator is positive and the denominator approaches 0 through negative values as \(x \to 2^-\).

19. Divide both the numerator and denominator by \(x^2\) (the highest power of \(x\) that occurs in the denominator).

\[
\lim_{x \to -\infty} \frac{x^3 + 5x}{2x^2 - x^2 + 4} = \lim_{x \to -\infty} \frac{\frac{x^3}{x^2} + \frac{5x}{x^2}}{\frac{2x^2}{x^2} - \frac{x^2}{x^2} + \frac{4}{x^2}} = \lim_{x \to -\infty} \frac{1 + \frac{5}{x}}{2 - \frac{1}{x} + \frac{4}{x^2}} = \lim_{x \to -\infty} \left( \frac{1 + \frac{5}{x^2}}{2 - \frac{1}{x} + \frac{4}{x^3}} \right)
\]

\[
= \lim_{x \to -\infty} \frac{1 + 5}{2 - 1 + 4} = \frac{1 + 5(0)}{2 - 0 + 4(0)} = \frac{1}{2}
\]

20. \( \lim_{t \to \infty} \frac{t^2 + 2}{t^2 + t^2 - 1} = \lim_{t \to \infty} \frac{(t^2 + 2)/t^2}{(t^2 + t^2 - 1)/t^2} = \lim_{t \to \infty} \frac{1/t + 2/t^2}{1 + 1/t - 1/t^2} = \frac{0 + 0}{1 + 0 - 0} = 0
\)

21. \( \lim_{t \to \infty} \frac{\sqrt{t} + t^2}{2t - t^2} = \lim_{t \to \infty} \frac{(\sqrt{t} + t^2)/t^2}{(2t - t^2)/t^2} = \lim_{t \to \infty} \frac{1/t^{2/2} + 1}{2/t - 1} = \frac{0 + 1}{0 - 1} = -1
\)

22. \( \lim_{t \to \infty} \frac{t - t^{\sqrt{t}}}{2t^{3/2} + 3t - 5} = \lim_{t \to \infty} \frac{(t - t^{\sqrt{t}})/t^{3/2}}{(2t^{3/2} + 3t - 5)/t^{3/2}} = \lim_{t \to \infty} \frac{1/t^{1/2} - 1}{2 + 3/t^{1/2} - 5/t^{1/2}} = \frac{0 - 1}{2 + 0 - 0} = -\frac{1}{2}
\)

23. \( \lim_{x \to \infty} \frac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)} = \lim_{x \to \infty} \frac{(2x^2 + 1)^2/x^4}{((x - 1)^2(x^2 + x))/x^4} = \lim_{x \to \infty} \frac{[(2x^2 + 1)/x^2]^2}{[(x^2 - 2x + 1)/x^2][(x^2 + x)/x^2]} = \lim_{x \to \infty} \frac{2 + x^2}{(2 + 0)(1 + 1/x)} = \frac{2 + 0^2}{(2 - 0)(1 + 0)} = 4
\)

24. \( \lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}} = \lim_{x \to \infty} \frac{\frac{x + 2}{x}}{\sqrt{9 + 1/x^2}} = \lim_{x \to \infty} \frac{1 + 2/x}{\sqrt{9 + 1/x^2}} = \frac{1 + 0}{\sqrt{9 + 0}} = \frac{1}{3}
\)
25. \[ \lim_{x \to \infty} \frac{\sqrt{9x^2 + x} - 3x}{\sqrt{9x^2 + x} + 3x} = \lim_{x \to \infty} \frac{(\sqrt{9x^2 + x} + 3x)(\sqrt{9x^2 + x} - 3x)}{\sqrt{9x^2 + x} + 3x} = \lim_{x \to \infty} \frac{(\sqrt{9x^2 + x} + 3x)(\sqrt{9x^2 + x} - 3x)}{\sqrt{9x^2 + x} + 3x} \] 
\[ = \lim_{x \to \infty} \frac{x}{\sqrt{9x^2 + x} + 3x} = \lim_{x \to \infty} \frac{1}{\sqrt{9 + 1/x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{12} = \frac{1}{6} \]

26. \[ \lim_{x \to \infty} \left( \frac{x^2 + ax - \sqrt{x^2 + bx}}{x / \sqrt{x^2 + bx}} \right) = \lim_{x \to \infty} \frac{(x^2 + ax - \sqrt{x^2 + bx})(\sqrt{x^2 + ax + \sqrt{x^2 + bx}})}{\sqrt{x^2 + ax + \sqrt{x^2 + bx}}} \] 
\[ = \lim_{x \to \infty} \frac{x^2 + ax}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + ax} + \sqrt{x^2 + bx}} \] 
\[ = \lim_{x \to \infty} \frac{a - b}{\sqrt{1 + a/x} + \sqrt{1 + b/x}} = \frac{a - b}{\sqrt{1 + 0} + \sqrt{1 + 0}} = \frac{a - b}{2} \]

27. \[ \lim_{x \to \infty} \frac{x^4 - 3x^2 + x}{x^2 - x + 2} = \lim_{x \to \infty} \frac{(x^4 - 3x^2 + x)/x^2}{(x^2 - x + 2)/x^2} \] 
\[ \text{[divide by the highest power of}\ x\ \text{in the denominator]} \] 
\[ \lim_{x \to \infty} \frac{x - 3/x + 1/x^2}{1 - 1/x^2 + 2/x^2} = \infty \] 

since the numerator increases without bound and the denominator approaches 1 as \( x \to \infty \).

28. Since \( 0 \leq \sin^2 x \leq 1 \), we have \( 0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \). Now \( \lim_{x \to \infty} 0 = 0 \) and \( \lim_{x \to \infty} \frac{1}{x^2} = 0 \), so by the Squeeze Theorem,
\[ \lim_{x \to \infty} \frac{\sin^2 x}{x^2} = 0. \]

29. \( \lim_{x \to \infty} \cos x \) does not exist because as \( x \) increases \( \cos x \) does not approach any one value, but oscillates between 1 and \( -1 \).

30. \[ \lim_{x \to \infty} \frac{x^2 - 2x + 3}{5 - 2x^2} = \lim_{x \to \infty} \frac{(x^2 - 2x + 3)/x^2}{(5 - 2x^2)/x^2} \] 
\[ \text{[divide by the highest power of}\ x\ \text{in the denominator]} \] 
\[ = \lim_{x \to \infty} \frac{x - 2/2 + 3/x}{5/x^2 - 2} = -\infty \] 
because \( x - 2/x + 3/x^2 \to \infty \) and \( 5/x^2 - 2 \to -2 \) as \( x \to \infty \).

31. \[ \lim_{x \to \infty} (x - \sqrt{x}) = \lim_{x \to \infty} \sqrt{x} (\sqrt{x} - 1) = \infty \] 
since \( \sqrt{x} \to \infty \) and \( \sqrt{x} - 1 \to \infty \) as \( x \to \infty \).

32. \[ \lim_{x \to \infty} (x^2 - x^4) = \lim_{x \to \infty} x^2 (1 - x^2) = -\infty \] 
since \( x^2 \to \infty \) and \( 1 - x^2 \to -\infty \).

33. \[ \lim_{x \to \infty} (x^4 + x^5) = \lim_{x \to \infty} x^5 (1 + 1/x) \] 
[factor out the largest power of \( x \)] 
\[ = -\infty \] 
because \( x^5 \to -\infty \) and \( 1/x + 1 \to 1 \) as \( x \to -\infty \).

Or: \[ \lim_{x \to -\infty} (x^4 + x^5) = \lim_{x \to -\infty} x^4 (1 + x) = -\infty. \]
34. (a) 

From the graph, it appears at first that there is only one horizontal asymptote, at $y \approx 0$, and a vertical asymptote at $x \approx 1.7$. However, if we graph the function with a wider and shorter viewing rectangle, we see that in fact there seem to be two horizontal asymptotes: one at $y \approx 0.5$ and one at $y \approx -0.5$. So we estimate that 

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \approx 0.5 \quad \text{and} \quad \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \approx -0.5$$

(b) $f(1000) \approx 0.4722$ and $f(10,000) \approx 0.4715$, so we estimate that $\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \approx 0.47$.

$f(-1000) \approx -0.4706$ and $f(-10,000) \approx -0.4713$, so we estimate that $\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \approx -0.47$.

(c) $\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\sqrt{2 + 1}/x^2}{3 - 5/x} \quad \text{[since $\sqrt{x^2} = x$ for $x > 0$]} = \frac{\sqrt{2}}{3} \approx 0.471404$.

For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the numerator by $x$, with $x < 0$, we get $\frac{1}{x} \sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}} \sqrt{2x^2 + 1} = -\sqrt{2 + 1/x^2}$. Therefore,

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + 1/x^2}}{3 - 5/x} = -\frac{\sqrt{2}}{3} \approx -0.471404.$$

35. $\lim_{x \to \infty} \frac{2x^2 + x - 1}{x^2 + x - 2} = \lim_{x \to \infty} \frac{2x^2 + x - 1}{x^2} = \lim_{x \to \infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}} = \lim_{x \to \infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}}$

$$= \frac{\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x}}{\lim_{x \to \infty} 1 + \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}} = \frac{2 + 0 - 0}{1 + 0 - 2(0)} = 2,$$

so $y = 2$ is a horizontal asymptote.

$$y = f(x) = \frac{2x^2 + x - 1}{x^2 + x - 2} = \frac{(2x - 1)(x + 1)}{(x + 2)(x - 1)},$$

so $\lim_{x \to 2} f(x) = \infty$.

$\lim_{x \to -2^+} f(x) = -\infty$, $\lim_{x \to -2^-} f(x) = -\infty$, and $\lim_{x \to 1^+} f(x) = \infty$. Thus, $x = -2$ and $x = 1$ are vertical asymptotes. The graph confirms our work.
36. \[ \lim_{x \to \infty} \frac{x - 9}{\sqrt{4x^2 + 3x + 2}} = \lim_{x \to \infty} \frac{1 - 9/x}{\sqrt{4 + (3/x) + (2/x^2)}} = \frac{1 - 0}{\sqrt{4 + 0 + 0}} = \frac{1}{2}. \]

Using the fact that \( \sqrt{x^2} = |x| = -x \) for \( x < 0 \), we divide the numerator by \(-x\) and the denominator by \( \sqrt{x^2} = \sqrt{(-x)^2} = x \).

Thus, \[ \lim_{x \to -\infty} \frac{x - 9}{\sqrt{4x^2 + 3x + 2}} = \lim_{x \to -\infty} \frac{-1 + 9}{\sqrt{4 + (3/x) + (2/x^2)}} = \frac{-1 + 0}{\sqrt{4 + 0 + 0}} = \frac{-1}{2}. \]

The horizontal asymptotes are \( y = \pm \frac{1}{2} \). The polynomial \( 4x^2 + 3x + 2 \) is positive for all \( x \), so the denominator never approaches zero, and thus there is no vertical asymptote.

37. (a) From the graph of \( f(x) = \sqrt{x^2 + x + 1} + x \), we estimate the value of \( \lim_{x \to -\infty} f(x) \) to be \(-0.5\).

(b) From the table, we estimate the limit to be \(-0.5\).

\[
\begin{array}{c|c}
 x & f(x) \\
-10,000 & -0.4999625 \\
-100,000 & -0.4999962 \\
-1,000,000 & -0.4999996 \\
\end{array}
\]

(c) \[ \lim_{x \to -\infty} (\sqrt{x^2 + x + 1} + x) = \lim_{x \to -\infty} (\sqrt{x^2 + x + 1} + x) \left[ \frac{\sqrt{x^2 + x + 1} - x}{\sqrt{x^2 + x + 1} - x} \right] = \lim_{x \to -\infty} \frac{x^2 + x + 1 - x^2}{x^2 + x + 1 - x} = \lim_{x \to -\infty} \frac{(x + 1)(1/x)}{(x^2 + x + 1 - x) - x} = \lim_{x \to -\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1} = \frac{1 + 0}{-\sqrt{1 + 0 + 0} - 1} = \frac{1}{2} \]

Note that for \( x < 0 \), we have \( \sqrt{x^2} = |x| = -x \), so when we divide the radical by \( x \), with \( x < 0 \), we get \[
\frac{1}{x} \sqrt{x^2 + x + 1} = -\frac{1}{x} \sqrt{x^2 + x + 1} = -\sqrt{1 + (1/x) + (1/x^2)}. \]
38. (a) \[ f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}, \]

From the graph of \( f(x) \), we estimate (to one decimal place) the value of \( \lim_{x \to -\infty} f(x) \) to be 1.4.

(c) \[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}}
\]

\[
= \lim_{x \to -\infty} \frac{(3x^2 + 8x + 6) - (3x^2 + 3x + 1)}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}}
\]

\[
= \lim_{x \to -\infty} \frac{5 + 5/x}{\sqrt{3} + \sqrt{3} + 3 + 3/x + 1/x^2}
\]

\[
= \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376
\]

39. From the graph, it appears \( y = 1 \) is a horizontal asymptote.

\[
\lim_{x \to -\infty} \frac{3x^3 + 500x^2}{x^2 + 500x^2 + 100x + 2000} = \lim_{x \to -\infty} \frac{3x^3 + 500x^2}{x^3} = \lim_{x \to -\infty} \frac{3 + 500/x}{1 + 500/x + 100/x^2 + 2000/x^3}
\]

\[
= \frac{3 + 0}{1 + 0 + 0 + 0} = 3, \text{ so } y = 3 \text{ is a horizontal asymptote.}
\]

The discrepancy can be explained by the choice of the viewing window. Try \([-100,000, 100,000]\) by \([-1, 4]\) to get a graph that lends credibility to our calculation that \( y = 3 \) is a horizontal asymptote.

40. Since the function has vertical asymptotes \( x = 1 \) and \( x = 3 \), the denominator of the rational function we are looking for must have factors \((x - 1)\) and \((x - 3)\). Because the horizontal asymptote is \( y = 1 \), the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is \( f(x) = \frac{x^2}{(x - 1)(x - 3)} \).
41. Let’s look for a rational function.

1. \[ \lim_{x \to \pm \infty} f(x) = 0 \quad \Rightarrow \quad \text{degree of numerator} < \text{degree of denominator} \]

2. \[ \lim_{x \to 0} f(x) = -\infty \quad \Rightarrow \quad \text{there is a factor of } x^2 \text{ in the denominator (not just } x, \text{ since that would produce a sign change at } x = 0), \text{ and the function is negative near } x = 0. \]

3. \[ \lim_{x \to 3^-} f(x) = \infty \text{ and } \lim_{x \to 3^+} f(x) = -\infty \quad \Rightarrow \quad \text{vertical asymptote at } x = 3; \text{ there is a factor of } (x - 3) \text{ in the denominator.} \]

4. \[ f(2) = 0 \quad \Rightarrow \quad 2 \text{ is an } x \text{-intercept, there is at least one factor of } (x - 2) \text{ in the numerator.} \]

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us

\[ f(x) = \frac{2-x}{x^2(x-3)} \text{ as one possibility.} \]

42. (a) If \( t = \frac{1}{x} \), then \[ \lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{t \to 0^+} \frac{1}{t} \sin t = \lim_{t \to 0^+} \frac{\sin t}{t} = 1. \]

(b) If \( t = \frac{1}{x} \), then \[ \lim_{x \to \infty} \sqrt{x} \sin \frac{1}{x} = \lim_{t \to 0^+} \frac{1}{\sqrt{t}} \sin t = \lim_{t \to 0^+} \frac{t \sin t}{t} = \lim_{t \to 0^+} \sqrt{t} \cdot \lim_{t \to 0^+} \frac{\sin t}{t} = 0 \cdot 1 = 0. \]

43. (a) We must first find the function \( f \). Since \( f \) has a vertical asymptote \( x = 4 \) and \( x \)-intercept \( x = 1 \), \( x - 4 \) is a factor of the denominator and \( x - 1 \) is a factor of the numerator. There is a removable discontinuity at \( x = -1 \), so \( x - (-1) = x + 1 \) is a factor of both the numerator and denominator. Thus, \( f \) now looks like this: \[ f(x) = \frac{a(x-1)(x+1)}{(x-4)(x+1)} \], where \( a \) is still to be determined. Then \[ \lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{a(x-1)(x+1)}{(x-4)(x+1)} = \lim_{x \to -1} \frac{a(x-1)}{x-4} = \frac{a(-1-1)}{(-1-4)} = \frac{2}{5}a, \text{ so } \frac{2}{5}a = 2, \text{ and } a = 5. \]

Thus \( f(x) = \frac{5(x-1)(x+1)}{(x-4)(x+1)} \) is a ratio of quadratic functions satisfying all the given conditions and

\[ f(0) = \frac{5(-1)(1)}{(-4)(1)} = \frac{5}{4}. \]

(b) \[ \lim_{x \to \infty} f(x) = 5 \lim_{x \to \infty} \frac{x^2 - 1}{x^2 - 3x - 4} = 5 \lim_{x \to \infty} \frac{(x^2/x^2) - (1/x^2)}{(x^2/x^2) - (3/x^2) - (4/x^2)} = 5 \frac{1 - 0}{1 - 0 - 0} = 5(1) = 5. \]
44. (a) In both viewing rectangles,
\[
\lim_{x \to -\infty} P(x) = \lim_{x \to -\infty} Q(x) = \infty \quad \text{and} \\
\lim_{x \to -\infty} P(x) = \lim_{x \to -\infty} Q(x) = -\infty.
\]
In the larger viewing rectangle, \( P \) and \( Q \) become less distinguishable.

(b) \[
\lim_{x \to -\infty} \frac{P(x)}{Q(x)} = \lim_{x \to -\infty} \frac{3x^5 - 5x^3 + 2x}{3x^5} = \lim_{x \to -\infty} \left(1 - \frac{5}{3} \cdot \frac{1}{x^2} + \frac{2}{3} \cdot \frac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \Rightarrow
\]
P and \( Q \) have the same end behavior.

45. Divide the numerator and the denominator by the highest power of \( x \) in \( Q(x) \).

(a) If \( \deg P < \deg Q \), then the numerator \( \to 0 \) but the denominator doesn’t. So \( \lim_{x \to -\infty} \frac{|P(x)|}{|Q(x)|} = 0 \).

(b) If \( \deg P > \deg Q \), then the numerator \( \to \pm \infty \) but the denominator doesn’t, so \( \lim_{x \to -\infty} \frac{|P(x)|}{|Q(x)|} = \pm \infty \)

(depending on the ratio of the leading coefficients of \( P \) and \( Q \)).

46.

(i) \( n = 0 \)  
(ii) \( n > 0 \) (n odd)  
(iii) \( n > 0 \) (n even)  
(iv) \( n < 0 \) (n odd)  
(v) \( n < 0 \) (n even)

From these sketches we see that

(a) \[
\lim_{x \to -0^+} x^n = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{if } n > 0 \\
\infty & \text{if } n < 0
\end{cases}
\]

(b) \[
\lim_{x \to -0^-} x^n = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{if } n > 0 \\
\infty & \text{if } n < 0, \ n \text{ odd} \\
\infty & \text{if } n < 0, \ n \text{ even}
\end{cases}
\]

(c) \[
\lim_{x \to \infty} x^n = \begin{cases} 
1 & \text{if } n = 0 \\
\infty & \text{if } n > 0 \\
0 & \text{if } n < 0
\end{cases}
\]

(d) \[
\lim_{x \to -\infty} x^n = \begin{cases} 
1 & \text{if } n = 0 \\
-\infty & \text{if } n > 0, \ n \text{ odd} \\
\infty & \text{if } n > 0, \ n \text{ even} \\
0 & \text{if } n < 0
\end{cases}
\]

47. \[
\lim_{x \to \infty} \frac{4x - 1}{x} = \lim_{x \to \infty} \left(4 - \frac{1}{x}\right) = 4 \quad \text{and} \quad \lim_{x \to \infty} \frac{4x^2 + 3x}{x^2} = \lim_{x \to \infty} \left(4 + \frac{3}{x}\right) = 4.
\]
Therefore, by the Squeeze Theorem,
\[
\lim_{x \to \infty} f(x) = 4.
\]

48. \[
\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_q}{\sqrt{1 - v^2/c^2}}.
\]
As \( v \to c^- \), \( \sqrt{1 - v^2/c^2} \to 0^+ \), and \( m \to \infty \).
49. (a) After $t$ minutes, $25t$ liters of brine with $30$ g of salt per liter has been pumped into the tank, so it contains

$$(5000 + 25t) \text{ liters of water and } 25t \cdot 30 = 750t \text{ grams of salt. Therefore, the salt concentration at time } t \text{ will be}$$

$$C(t) = \frac{750t}{5000 + 25t} = \frac{30t}{200 + 5t} \text{ g} \text{ L}^{-1}.$$  

(b) $\lim_{t \to \infty} C(t) = \lim_{t \to \infty} \frac{30t}{200 + 5t} = \lim_{t \to \infty} \frac{30t/t}{200/t + 5/t} = \frac{30}{0 + 1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

50. (a) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{4x^2 - 5x}{2x^2 + 1} = \lim_{x \to \infty} \frac{4 - 5/x}{2 + 1/x^2} = \frac{4}{2} = 2$

(b) $f(x) = 1.9 \Rightarrow x \approx 25.374$, so $f(x) > 1.9$ when $x > N = 25.4$.

$f(x) = 1.99 \Rightarrow x \approx 250.3974$, so $f(x) > 1.99$ when $x > N = 250.4$. 

![Graph of function with limits](image)
51. \[
\frac{1}{(x + 3)^4} > 10,000 \iff (x + 3)^4 < \frac{1}{10,000} \iff |x + 3| < \frac{1}{\sqrt[4]{10,000}} \iff |x - (-3)| < \frac{1}{10}
\]

52. Given \( M > 0 \), we need \( \delta > 0 \) such that \( 0 < |x + 3| < \delta \implies \frac{1}{(x + 3)^4} > M \). Now \( \frac{1}{(x + 3)^4} > M \iff (x + 3)^4 < \frac{1}{M} \iff |x + 3| < \frac{1}{\sqrt[4]{M}} \). So take \( \delta = \frac{1}{\sqrt[4]{M}} \). Then \( 0 < |x + 3| < \delta = \frac{1}{\sqrt[4]{M}} \implies \frac{1}{(x + 3)^4} > M \), so
\[
\lim_{x \to -3} \frac{1}{(x + 3)^4} = \infty.
\]

53. Let \( N < 0 \) be given. Then, for \( x < -1 \), we have \( \frac{5}{(x + 1)^2} < N \iff \frac{5}{N} < (x + 1)^2 \iff \sqrt{\frac{5}{N}} < x + 1 \). Let \( \delta = -\sqrt{\frac{5}{N}} \). Then \( -1 - \delta < x < -1 \implies \sqrt{\frac{5}{N}} < x + 1 < 0 \implies \frac{5}{(x + 1)^2} < N \), so \( \lim_{x \to -1} \frac{5}{(x + 1)^2} = -\infty \).

54. For \( \varepsilon = 0.5 \), we must find \( N \) such that whenever \( x \geq N \), we have \( |\frac{\sqrt{4x^2 + 1}}{x + 1} - 2| < 0.5 \iff 1.5 < \frac{\sqrt{4x^2 + 1}}{x + 1} < 2.5 \).

We graph the three parts of this inequality on the same screen, and find that it holds whenever \( x > 2.82 \). So we choose \( N = 3 \) (or any larger number). For \( \varepsilon = 0.1 \), we must have \( 1.9 \sqrt{4x^2 + 1}/x + 1 < 2.1 \), and the graphs show that this holds whenever \( x > 18.9 \). So we choose \( N = 19 \) (or any larger number).

\[
\begin{array}{c}
\text{Graph 1} \\
\text{Graph 2} \\
\text{Graph 3}
\end{array}
\]

55. Let \( g(x) = \frac{3x^2 + 1}{2x^2 + x + 1} \) and \( f(x) = |g(x) - 1.5| \). Note that \( \lim_{x \to \infty} g(x) = \frac{3}{2} \) and \( \lim_{x \to \infty} f(x) = 0 \). We are interested in finding the \( x \)-value at which \( f(x) < 0.05 \). From the graph, we find that \( x \approx 14.804 \), so we choose \( N = 15 \) (or any larger number).
56. We need \( N \) such that \[ \frac{2x + 1}{\sqrt{x + 1}} > 100 \] whenever \( x \geq N \). From the graph, we see that this inequality holds for \( x \geq 2500 \). So we choose \( N = 2500 \) (or any larger number).

\[
\begin{align*}
57. \text{(a) } & \frac{1}{x^2} < 0.0001 \iff x^2 > 1/0.0001 = 10000 \iff x > 100 \quad (x > 0) \\
& \text{(b) If } \varepsilon > 0 \text{ is given, then } 1/x^2 < \varepsilon \iff x^2 > 1/\varepsilon \iff x > 1/\sqrt{\varepsilon}. \text{ Let } N = 1/\sqrt{\varepsilon}.
\end{align*}
\]

Then \( x > N \implies x > \frac{1}{\sqrt{\varepsilon}} \implies \left| \frac{1}{x^2} - 0 \right| = \frac{1}{x^2} < \varepsilon, \text{ so } \lim_{x \to \infty} \frac{1}{x^2} = 0. \)

58. Given \( M > 0 \), we need \( N > 0 \) such that \( x > N \implies x^2 > M \). Now \( x^2 > M \implies x > \sqrt{M} \), so take \( N = \sqrt{M} \). Then \( x > N = \sqrt{M} \implies x^2 > M, \text{ so } \lim_{x \to \infty} x^2 = \infty. \)

59. Suppose that \( \lim_{x \to -\infty} f(x) = L \). Then for every \( \varepsilon > 0 \) there is a corresponding positive number \( N \) such that \( |f(x) - L| < \varepsilon \) whenever \( x > N \). If \( t = 1/x \), then \( x > N \iff 0 < 1/x < 1/N \iff 0 < t < 1/N \). Thus, for every \( \varepsilon > 0 \) there is a corresponding \( \delta > 0 \) (namely \( 1/N \)) such that \( |f(1/t) - L| < \varepsilon \) whenever \( 0 < t < \delta \). This proves that

\[
\lim_{t \to 0^+} f(1/t) = L = \lim_{x \to -\infty} f(x).
\]

Now suppose that \( \lim_{x \to -\infty} f(x) = L \). Then for every \( \varepsilon > 0 \) there is a corresponding negative number \( N \) such that \( |f(x) - L| < \varepsilon \) whenever \( x < N \). If \( t = 1/x \), then \( x < N \iff 1/N < 1/x < 0 \iff 1/N < t < 0 \). Thus, for every \( \varepsilon > 0 \) there is a corresponding \( \delta > 0 \) (namely \(-1/N\)) such that \( |f(1/t) - L| < \varepsilon \) whenever \(-\delta < t < 0 \). This proves that

\[
\lim_{t \to 0^-} f(1/t) = L = \lim_{x \to -\infty} f(x).
\]
1. (a) A function $f$ is a rule that assigns to each element $x$ in a set $D$ exactly one element, called $f(x)$, in a set $E$. The set $D$ is called the domain of the function. The range of $f$ is the set of all possible values of $f(x)$ as $x$ varies throughout the domain.

(b) If $f$ is a function with domain $D$, then its graph is the set of ordered pairs $\{(x, f(x)) \mid x \in D\}$.

(c) Use the Vertical Line Test on page 4.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

   Verbally: An assignment of students to chairs in a classroom (a description in words)
   Numerically: A tax table that assigns an amount of tax to an income (a table of values)
   Visually: A graphical history of the Dow Jones average (a graph)
   Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) If a function $f$ satisfies $f(-x) = f(x)$ for every number $x$ in its domain, then $f$ is called an even function. If the graph of a function is symmetric with respect to the $y$-axis, then $f$ is even. Examples of an even function: $f(x) = x^2$, $f(x) = x^4 + x^2$, $f(x) = |x|$, $f(x) = \cos x$.

(b) If a function $f$ satisfies $f(-x) = -f(x)$ for every number $x$ in its domain, then $f$ is called an odd function. If the graph of a function is symmetric with respect to the origin, then $f$ is odd. Examples of an odd function: $f(x) = x^3$, $f(x) = x^5 + x^3$, $f(x) = \sqrt[3]{x}$, $f(x) = \sin x$.

4. A function $f$ is called increasing on an interval $I$ if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in $I$.

5. A mathematical model is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^5$

(c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

(e) Polynomial of degree 5: $f(x) = x^5 + 2$

(f) Rational function: $f(x) = \frac{x}{x + 2}$, $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials
9. (a) The domain of \( f + g \) is the intersection of the domain of \( f \) and the domain of \( g \); that is, \( A \cap B \).
   
   (b) The domain of \( fg \) is also \( A \cap B \).
   
   (c) The domain of \( f/g \) must exclude values of \( x \) that make \( g \) equal to 0; that is, \( \{ x \in A \cap B \mid g(x) \neq 0 \} \).

10. Given two functions \( f \) and \( g \), the composite function \( f \circ g \) is defined by \( (f \circ g)(x) = f(g(x)) \). The domain of \( f \circ g \) is the set of all \( x \) in the domain of \( g \) such that \( g(x) \) is in the domain of \( f \).
11. (a) If the graph of \( f \) is shifted 2 units upward, its equation becomes \( y = f(x) + 2 \).
(b) If the graph of \( f \) is shifted 2 units downward, its equation becomes \( y = f(x) - 2 \).
(c) If the graph of \( f \) is shifted 2 units to the right, its equation becomes \( y = f(x - 2) \).
(d) If the graph of \( f \) is shifted 2 units to the left, its equation becomes \( y = f(x + 2) \).
(e) If the graph of \( f \) is reflected about the \( x \)-axis, its equation becomes \( y = -f(x) \).
(f) If the graph of \( f \) is reflected about the \( y \)-axis, its equation becomes \( y = f(-x) \).
(g) If the graph of \( f \) is stretched vertically by a factor of 2, its equation becomes \( y = 2f(x) \).
(h) If the graph of \( f \) is shrunk vertically by a factor of 2, its equation becomes \( y = \frac{1}{2}f(x) \).
(i) If the graph of \( f \) is stretched horizontally by a factor of 2, its equation becomes \( y = f\left(\frac{1}{2}x\right) \).
(j) If the graph of \( f \) is shrunk horizontally by a factor of 2, its equation becomes \( y = f(2x) \).

12. (a) \( \lim_{x \to 0} f(x) = L \): See Definition 1.3.1 and Figures 1 and 2 in Section 1.3.
(b) \( \lim_{x \to +\infty} f(x) = L \): See the paragraph after Definition 1.3.2 and Figure 9(b) in Section 1.3.
(c) \( \lim_{x \to -\infty} f(x) = L \): See Definition 1.3.2 and Figure 9(a) in Section 1.3.
(d) \( \lim_{x \to 0} f(x) = \infty \): See Definition 1.6.1 and Figure 2 in Section 1.6.
(e) \( \lim_{x \to -\infty} f(x) = \infty \): See Definition 1.6.3 and Figure 8 in Section 1.6.

13. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at \( x = 2 \).

![Graphs showing the limit fails to exist](image1.png)

The left- and right-hand limits are not equal.

![Graph showing an infinite discontinuity](image2.png)

There is an infinite discontinuity.

![Graph showing an infinite number of oscillations](image3.png)

There are an infinite number of oscillations.

14. (a) – (g) See the statements of Limit Laws 1–6 and 11 in Section 1.4.

15. See Theorem 4 in Section 1.4.
16. (a) A function $f$ is continuous at a number $a$ if $f(x)$ approaches $f(a)$ as $x$ approaches $a$; that is, \( \lim_{x \to a} f(x) = f(a) \).

(b) A function $f$ is continuous on the interval $(-\infty, \infty)$ if $f$ is continuous at every real number $a$. The graph of such a function has no breaks and every vertical line crosses it.

17. See Theorem 1.5.9.

18. (a) See Definition 1.6.2 and Figures 2–4 in Section 1.6.

(b) See Definition 1.6.4 and Figures 8 and 9 in Section 1.6.
Chapter 01-Exercises

1. (a) When \( x = 2, y \approx 2.7 \). Thus, \( f(2) \approx 2.7 \).
   
   (b) \( f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6 \).
   
   (c) The domain of \( f \) is \(-6 \leq x \leq 6\), or \([-6, 6]\).
   
   (d) The range of \( f \) is \(-4 \leq y \leq 4\), or \([-4, 4]\).
   
   (e) \( f \) is increasing on \([-4, 4]\), that is, on \(-4 \leq x \leq 4\).
   
   (f) \( f \) is odd since its graph is symmetric about the origin.

2. (a) This curve is not the graph of a function of \( x \) since it fails the Vertical Line Test.
   
   (b) This curve is the graph of a function of \( x \) since it passes the Vertical Line Test. The domain is \([-3, 3]\) and the range is \([-2, 3]\).

3. \( f(x) = \frac{2}{3x - 1} \).
   
   Domain: \( 3x - 1 \neq 0 \quad \Rightarrow \quad 3x \neq 1 \quad \Rightarrow \quad x \neq \frac{1}{3} \). \( \quad D = (-\infty, \frac{1}{3}) \cup (\frac{1}{3}, \infty) \)
   
   Range: all reals except 0 (\( y = 0 \) is the horizontal asymptote for \( f \)). \( \quad R = (-\infty, 0) \cup (0, \infty) \)

4. \( g(x) = \sqrt{16 - x^4} \).
   
   Domain: \( 16 - x^4 \geq 0 \quad \Rightarrow \quad x^4 \leq 16 \quad \Rightarrow \quad |x| \leq \sqrt[4]{16} \quad \Rightarrow \quad |x| \leq 2 \). \( \quad D = [-2, 2] \)
   
   Range: \( y \geq 0 \) and \( y \leq \sqrt{16} \quad \Rightarrow \quad 0 \leq y \leq 4 \). \( \quad R = [0, 4] \)

5. \( y = 1 + \sin x \).
   
   Domain: \( \mathbb{R} \)
   
   Range: \( -1 \leq \sin x \leq 1 \quad \Rightarrow \quad 0 \leq 1 + \sin x \leq 2 \quad \Rightarrow \quad 0 \leq y \leq 2 \).

6. \( y = \tan 2x \).
   
   Domain: \( 2x \neq \frac{\pi}{2} + \pi n \quad \Rightarrow \quad x \neq \frac{\pi}{4} + \frac{\pi}{2} n \).
   
   Range: the tangent function takes on all real values, so the range is \( \mathbb{R} \).

7. (a) To obtain the graph of \( y = f(x) + 8 \), we shift the graph of \( y = f(x) \) up 8 units.
   
   (b) To obtain the graph of \( y = f(x) + 8 \), we shift the graph of \( y = f(x) \) left 8 units.
   
   (c) To obtain the graph of \( y = 1 + 2f(x) \), we stretch the graph of \( y = f(x) \) vertically by a factor of 2, and then shift the resulting graph 1 unit upward.
   
   (d) To obtain the graph of \( y = f(x - 2) - 2 \), we shift the graph of \( y = f(x) \) right 2 units (for the “-2” inside the parentheses), and then shift the resulting graph 2 units downward.
   
   (e) To obtain the graph of \( y = -f(x) \), we reflect the graph of \( y = f(x) \) about the x-axis.
   
   (f) To obtain the graph of \( y = 3 - f(x) \), we reflect the graph of \( y = f(x) \) about the x-axis, and then shift the resulting graph 3 units upward.
8. (a) To obtain the graph of \( y = f(x - 8) \), we shift the graph of \( y = f(x) \) right 8 units.

(b) To obtain the graph of \( y = -f(x) \), we reflect the graph of \( y = f(x) \) about the x-axis.

(c) To obtain the graph of \( y = 2 - f(x) \), we reflect the graph of \( y = f(x) \) about the x-axis, and then shift the resulting graph 2 units upward.

(d) To obtain the graph of \( y = \frac{1}{2}f(x) - 1 \), we shrink the graph of \( y = f(x) \) by a factor of 2, and then shift the resulting graph 1 unit downward.

9. \( y = -\sin 2x \): Start with the graph of \( y = \sin x \), compress horizontally by a factor of 2, and reflect about the x-axis.

10. \( y = (x - 2)^2 \): Start with the graph of \( y = x^2 \) and shift 2 units to the right.
11. \( y = 1 + \frac{1}{2}x^3 \): Start with the graph of \( y = x^3 \), compress vertically by a factor of 2, and shift 1 unit upward.

12. \( y = 2 - \sqrt{x} \):  
   Start with the graph of \( y = \sqrt{x} \),  
   reflect about the \( x \)-axis, and shift 2 units upward.

13. \( f(x) = \frac{1}{x + 2} \):  
   Start with the graph of \( f(x) = 1/x \)  
   and shift 2 units to the left.

14. \( f(x) = \begin{cases} 
1 + x & \text{if } x < 0 \\
1 + x^2 & \text{if } x \geq 0
\end{cases} \)
   On \( (-\infty, 0) \), graph \( y = 1 + x \) (the line with slope 1 and \( y \)-intercept 1) 
   with open endpoint \( (0, 1) \).
   On \( [0, \infty) \), graph \( y = 1 + x^2 \) (the rightmost half of the parabola \( y = x^2 \) 
   shifted 1 unit upward) with closed endpoint \( (0, 1) \).

15. (a) The terms of \( f \) are a mixture of odd and even powers of \( x \), so \( f \) is neither even nor odd.

   (b) The terms of \( f \) are all odd powers of \( x \), so \( f \) is odd.

   (c) \( f(-x) = \cos((-x)^2) = \cos(x^2) = f(x) \), so \( f \) is even.

   (d) \( f(-x) = 1 + \sin(-x) = 1 - \sin x \). Now \( f(-x) \neq f(x) \) and \( f(-x) \neq -f(x) \), so \( f \) is neither even nor odd.
16. For the line segment from \((-2, 2)\) to \((-1, 0)\), the slope is \(\frac{0 - 2}{-1 + 2} = -2\), and an equation is \(y - 0 = -2(x + 1)\) or, equivalently, \(y = -2x - 2\). The circle has equation \(x^2 + y^2 = 1\); the top half has equation \(y = \sqrt{1-x^2}\) (we have solved for positive \(y\)). Thus, \(f(x) = \begin{cases} -2x - 2 & \text{if } -2 \leq x \leq -1 \\ \sqrt{1-x^2} & \text{if } -1 < x \leq 1 \end{cases}\)

17. \(f(x) = \sqrt{x}, D = [0, \infty); g(x) = \sin x, D = \mathbb{R}\).
   (a) \((f \circ g)(x) = f(g(x)) = f(\sin x) = \sqrt{\sin x}\). For \(\sqrt{\sin x}\) to be defined, we must have \(\sin x \geq 0 \iff x \in [0, \pi], [2\pi, 3\pi], [-2\pi, -\pi], [4\pi, 5\pi], [-4\pi, -3\pi], \ldots\), so \(D = \{x \mid x \in [2n\pi, \pi + 2n\pi]\}, \text{ where } n \text{ is an integer}\).
   (b) \((g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sin \sqrt{x}\). \(x\) must be greater than or equal to 0 for \(\sqrt{x}\) to be defined, so \(D = [0, \infty)\).
   (c) \((f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt{x}. D = [0, \infty)\).
   (d) \((g \circ g)(x) = g(g(x)) = g(\sin x) = \sin(\sin x). D = \mathbb{R}\).

18. Let \(h(x) = x + \sqrt{x}, g(x) = \sqrt{x}, \text{ and } f(x) = 1/x\). Then \((f \circ g \circ h)(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = F(x)\).

19. (a) (i) \(\lim_{x \to 2^+} f(x) = 3\)  
   (ii) \(\lim_{x \to 3^-} f(x) = 0\)
   (iii) \(\lim_{x \to -3^-} f(x)\) does not exist since the left and right limits are not equal. (The left limit is -2.)
   (iv) \(\lim_{x \to 4} f(x) = 2\)
   (v) \(\lim_{x \to 0} f(x) = \infty\)
   (vi) \(\lim_{x \to -2^-} f(x) = -\infty\)
   (vii) \(\lim_{x \to 0} f(x) = 4\)
   (viii) \(\lim_{x \to -\infty} f(x) = -1\)
   (b) The equations of the horizontal asymptotes are \(y = -1\) and \(y = 4\).
   (c) The equations of the vertical asymptotes are \(x = 0\) and \(x = 2\).
   (d) \(f\) is discontinuous at \(-3, 0, 2, \text{ and } 4\). The discontinuities are jump, infinite, infinite, and removable, respectively.

20. \(\lim_{x \to -\infty} f(x) = -2, \quad \lim_{x \to -3} f(x) = 0, \quad \lim_{x \to -2} f(x) = \infty, \quad \lim_{x \to -2} f(x) = -\infty, \quad \lim_{x \to 2^+} f(x) = 2, \quad f\) is continuous from the right at 3

21. \(\lim_{x \to 0} \cos(x + \sin x) = \cos \left(\lim_{x \to 0} (x + \sin x)\right) \quad \text{[by Theorem 1.5.7]} = \cos 0 = 1\)
22. Since rational functions are continuous, \( \lim_{x \to 3} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{3^2 - 9}{3^2 + 2(3) - 3} = \frac{0}{12} = 0. \)

23. \( \lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \to -3} \frac{(x + 3)(x - 3)}{(x + 3)(x - 1)} = \lim_{x \to -3} \frac{x - 3}{x - 1} = \frac{-3 - 3}{-3 - 1} = \frac{-6}{-4} = \frac{3}{2} \)

24. \( \lim_{x \to 1^+} \frac{x^2 - 9}{x^2 + 2x - 3} = -\infty \) since \( x^2 + 2x - 3 \to 0^+ \) as \( x \to 1^+ \) and \( \frac{x^2 - 9}{x^2 + 2x - 3} < 0 \) for \( 1 < x < 3 \).

25. \( \lim_{h \to 0} \frac{(h - 1)^2 + 1}{h} = \lim_{h \to 0} \frac{(h - 1)^2 - h^2 + 3h - 1}{h} = \lim_{h \to 0} \frac{h^2 - 3h^2 + 3h}{h} = \lim_{h \to 0} (h^2 - 3h + 3) = 3 \)

Another solution: Factor the numerator as a sum of two cubes and then simplify.

\( \lim_{h \to 0} \frac{(h - 1)^2 + 1}{h} = \lim_{h \to 0} \frac{(h - 1)^2 + 1}{h} = \lim_{h \to 0} \frac{[(h - 1)^2 - 1(h - 1) + 1^3]}{h} \)

\( = \lim_{h \to 0} \frac{[(h - 1)^2 - h + 2]}{h} = 1 - 0 + 2 = 3 \)

26. \( \lim_{t \to 2} \frac{t^2 - 4}{t^2 - 8} = \lim_{t \to 2} \frac{(t + 2)(t - 2)}{(t - 2)(t^2 + 2t + 4)} = \lim_{t \to 2} \frac{t + 2}{t^2 + 2t + 4} = \frac{4 + 2}{4 + 4 + 4} = \frac{5}{12} = \frac{1}{3} \)

27. \( \lim_{r \to 0} \frac{\sqrt[4]{r}}{(r - 9)^4} = \infty \) since \( (r - 9)^4 \to 0^+ \) as \( r \to 9 \) and \( \frac{\sqrt[4]{r}}{(r - 9)^4} > 0 \) for \( r \neq 9 \).

28. \( \lim_{v \to 4^+} \frac{4 - v}{|4 - v|} = \lim_{v \to 4^+} \frac{4 - v}{(4 - v)} = \lim_{v \to 4^+} \frac{1}{-1} = -1 \)

29. \( \lim_{s \to 16} \frac{4 - \sqrt{s}}{s - 16} = \lim_{s \to 16} \frac{4 - \sqrt{s}}{(\sqrt{s} - 4)(\sqrt{s} + 4)} = \lim_{s \to 16} \frac{-1}{\sqrt{s} + 4} = \frac{-1}{\sqrt{16} + 4} = \frac{-1}{8} \)

30. \( \lim_{v \to 2} \frac{v^2 + 2v - 8}{v^4 - 16} = \lim_{v \to 2} \frac{(v + 4)(v - 2)}{(v + 2)(v - 2)(v^2 + 4)} = \lim_{v \to 2} \frac{v + 4}{(v + 2)(v^2 + 4)} = \frac{2 + 4}{(2 + 2)(2^2 + 4)} = \frac{3}{16} \)

31. \( \lim_{x \to \infty} \frac{1 + 2x - x^2}{1 - x + 2x^2} = \lim_{x \to \infty} \frac{(1 + 2x - x^2)/x^2}{(1 - x + 2x^2)/x^2} = \lim_{x \to \infty} \frac{1/x^2 + 2/x - 1}{1/x^2 - 1/x + 2} = \frac{0 + 0 - 1}{0 - 0 + 2} = -\frac{1}{2} \)

32. \( \lim_{x \to \infty} \frac{1 - 2x^2 - x^4}{5 + x - 3x^2} = \lim_{x \to \infty} \frac{(1 - 2x^2 - x^4)/x^4}{(5 + x - 3x^2)/x^4} = \lim_{x \to \infty} \frac{1/x^4 - 2/x^2 - 1}{5/x^4 + 1/x^2 - 3} = \frac{0 - 0 - 1}{0 + 0 - 3} = -\frac{1}{3} = \frac{1}{3} \)
33. \[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 4x + 1} - x}{1} = \lim_{x \to \infty} \frac{\sqrt{x^2 + 4x + 1} - x}{\sqrt{x^2 + 4x + 1} + x} = \lim_{x \to \infty} \frac{(4x + 1)/x}{\sqrt{x^2 + 4x + 1} + x} = \lim_{x \to \infty} \frac{4 + 1/x}{\sqrt{1 + 4/x + 1/x^2} + 1} = \frac{4 + 0}{\sqrt{1 + 0 + 0} + 1} = \frac{4}{2} = 2
\]

34. \[
\lim_{x \to -1} \left( \frac{1}{x - 1} + \frac{1}{x^2 - 3x + 2} \right) = \lim_{x \to -1} \left[ \frac{1}{x - 1} + \frac{1}{(x - 1)(x - 2)} \right] = \lim_{x \to -1} \left[ \frac{x - 2}{(x - 1)(x - 2)} + \frac{1}{(x - 1)(x - 2)} \right] = \lim_{x \to -1} \left[ \frac{x - 1}{(x - 1)(x - 2)} \right] = \lim_{x \to -1} \frac{1}{x - 2} = \frac{1}{-2} = -1
\]

35. \[
\lim_{x \to 0} \frac{\cot 2x}{\csc x} = \lim_{x \to 0} \frac{\cos 2x \sin x}{\sin 2x} = \lim_{x \to 0} \cos 2x \left[ \frac{\sin x}{\sin 2x} \right] = \lim_{x \to 0} \cos 2x \left[ \frac{\lim_{x \to 0} \sin x}{\lim_{x \to 0} \sin 2x} \right] = 1 \cdot \frac{1}{2} = \frac{1}{2}
\]

36. \[
\lim_{t \to 0} \frac{t^2}{\tan^2 2t} = \lim_{t \to 0} \frac{t^2 \cos^2 2t}{\sin^2 2t} = \lim_{t \to 0} \cos^2 2t \cdot \frac{1}{\frac{\sin^2 2t}{(2t)^2}} = \lim_{t \to 0} \cos^2 2t \left( \frac{\lim_{t \to 0} \sin 2t}{2t} \right) = \frac{1}{8} \cdot 1 = \frac{1}{8}
\]

37. From the graph of \( y = \frac{\cos^2 x}{x^2} \), it appears that \( y = 0 \) is the horizontal asymptote and \( x = 0 \) is the vertical asymptote. Now \( 0 \leq \cos^2 x \leq 1 \Rightarrow \)
\[
0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2} \Rightarrow 0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2} \]
But \( \lim_{x \to \pm\infty} 0 = 0 \) and \( \lim_{x \to \pm\infty} \frac{1}{x^2} = 0 \), so by the Squeeze Theorem, \( \lim_{x \to \pm\infty} \frac{\cos^2 x}{x^2} = 0 \).

Thus, \( y = 0 \) is the horizontal asymptote. \( \lim_{x \to 0} \frac{\cos^2 x}{x^2} = \infty \) because \( \cos^2 x \to 1 \) and \( x^2 \to 0^+ \) as \( x \to 0 \), so \( x = 0 \) is the vertical asymptote.
38. From the graph of \( y = f(x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \), it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for \( f \), let's multiply and divide it by its conjugate.

\[
f_1(x) = \frac{\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \times \frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} = \frac{(x^2 + x + 1) - (x^2 - x)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}
\]

\[
= \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}
\]

Now \( \lim_{x \to \infty} f_1(x) = \lim_{x \to \infty} \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \)

\[
= \lim_{x \to \infty} \frac{2 + (1/x)}{\sqrt{1 + (1/x^2)} + \sqrt{1 - (1/x)}}
\]

[since \( \sqrt{x^2} = x \) for \( x > 0 \)]

\[
= \frac{2}{1 + 1} = 1,
\]

so \( y = 1 \) is a horizontal asymptote. For \( x < 0 \), we have \( \sqrt{x^2} = |x| = -x \), so when we divide the denominator by \( x \), with \( x < 0 \), we get

\[
\frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{x} = -\frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2}} = -\left[ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x^2}} + \sqrt{1 - \frac{1}{x}} \right]
\]

Therefore,

\[
\lim_{x \to -\infty} f_1(x) = \lim_{x \to -\infty} \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} = \lim_{x \to -\infty} \frac{2 + (1/x)}{\sqrt{1 + (1/x^2)} + \sqrt{1 - (1/x)}}
\]

\[
= \frac{2}{-(1 + 1)} = -1,
\]

so \( y = -1 \) is a horizontal asymptote.

The domain of \( f \) is \((-\infty, 0] \cup [1, \infty)\). As \( x \to 0^- \), \( f(x) \to 1 \), so

\( x = 0 \) is not a vertical asymptote. As \( x \to 1^+ \), \( f(x) \to \sqrt{3} \), so \( x = 1 \) is not a vertical asymptote and hence there are no vertical asymptotes.

39. Since \( 2x - 1 \leq f(x) \leq x^2 \) for \( 0 < x < 3 \) and \( \lim_{x \to 1} (2x - 1) = 1 = \lim_{x \to 1} x^2 \), we have \( \lim_{x \to 1} f(x) = 1 \) by the Squeeze Theorem.

40. Let \( f(x) = -x^2 \), \( g(x) = x^2 \cos(1/x^2) \) and \( h(x) = x^2 \). Then since \( |\cos(1/x^2)| \leq 1 \) for \( x \neq 0 \), we have

\[
f(x) \leq g(x) \leq h(x) \quad \text{for} \ x \neq 0,
\]

and so \( \lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \Rightarrow \lim_{x \to 0} g(x) = 0 \) by the Squeeze Theorem.

41. Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then \( |(14 - 5x) - 4| < \varepsilon \). But \( |(14 - 5x) - 4| < \varepsilon \) \( \iff \) \( |(14 - 5x) - 4| < \varepsilon \). Given \( \varepsilon > 0 \), we need \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then \( |(14 - 5x) - 4| < \varepsilon \). But \( |(14 - 5x) - 4| < \varepsilon \) \( \iff \) \( |(14 - 5x) - 4| < \varepsilon \). So if we choose \( \delta = \varepsilon/5 \), then \( 0 < |x - 2| < \delta \Rightarrow |(14 - 5x) - 4| < \varepsilon \). Thus, \( \lim_{x \to 2} (14 - 5x) = 4 \) by the definition of a limit.
42. Given \( \varepsilon > 0 \) we must find \( \delta > 0 \) so that if \( 0 < |x - 0| < \delta \), then \( \frac{1}{\sqrt{x}} - 0 < \varepsilon \cdot \text{Now} \cdot \frac{1}{\sqrt{x}} - 0 = \frac{1}{\sqrt{x}} < \varepsilon \Rightarrow |x| = \left| \frac{1}{\sqrt{x}} \right|^2 < \varepsilon^2. \text{So take} \cdot \delta = \varepsilon^2. \text{Then} \cdot 0 < |x - 0| = |x| < \varepsilon^2 \Rightarrow \frac{1}{\sqrt{x}} - 0 = \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{|x|}} < \sqrt{\varepsilon^2} = \varepsilon. \text{Therefore, by the definition of a limit,} \lim_{x \to 0} \frac{1}{\sqrt{x}} = 0. \)

43. If \( \varepsilon > 0 \) is given, then \( 1/x^4 < \varepsilon \iff x^4 > 1/\varepsilon \iff x > 1/\sqrt[4]{\varepsilon}. \text{Let} \cdot N = 1/\sqrt[4]{\varepsilon}. \)

Then \( x > N \Rightarrow x > \frac{1}{\sqrt[4]{\varepsilon}} \Rightarrow \left| \frac{1}{x^4} - 0 \right| = \frac{1}{x^4} < \varepsilon, \text{so} \lim_{x \to \infty} \frac{1}{x^4} = 0. \)

44. Given \( M > 0 \), we need \( \delta > 0 \) such that if \( 0 < x - 4 < \delta \), then \( 2/\sqrt{x-4} > M. \text{This is true} \iff \sqrt{x-4} < 2/M \iff x-4 < 4/M^2. \text{So if we choose} \cdot \delta = 4/M^2, \text{then} \cdot 0 < x - 4 < \delta \Rightarrow 2/\sqrt{x-4} > M. \text{So by the definition of a limit,} \lim_{x \to 4^+} (2/\sqrt{x-4}) = \infty. \)

45. (a) \( f(x) = \sqrt[3]{x} \) if \( x < 0 \), \( f(x) = 3 - x \) if \( 0 \leq x < 3 \), \( f(x) = (x - 3)^2 \) if \( x > 3. \)

(i) \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (3 - x) = 3 \)

(ii) \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = 0 \)

(iii) Because of (i) and (ii), \( \lim_{x \to 0} f(x) \) does not exist.

(iv) \( \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3 - x) = 0 \)

(v) \( \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3)^2 = 0 \)

(vi) Because of (iv) and (v), \( \lim_{x \to 3} f(x) = 0. \)

(b) \( f \) is discontinuous at 0 since \( \lim_{x \to 0} f(x) \) does not exist.

(c) \( f \) is discontinuous at 3 since \( f(3) \) does not exist.

46. (a) \( x^2 - 9 \) is continuous on \( \mathbb{R} \) since it is a polynomial and \( \sqrt{x} \) is continuous on \( [0, \infty) \) by Theorem 7 in Section 1.8, so the composition \( \sqrt{x^2 - 9} \) is continuous on \( \{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty) \) by Theorem 9. Note that \( x^2 - 2 \neq 0 \) on this set and so the quotient function \( g(x) = \frac{\sqrt{x^2 - 9}}{x^2 - 2} \) is continuous on its domain, \( (-\infty, -3] \cup [3, \infty) \) by Theorem 4.

(b) \( x^2 \) is continuous on \( \mathbb{R} \) since it is a polynomial and \( \cos x \) is also continuous on \( \mathbb{R} \), so the product \( x^2 \cos x \) is continuous on \( \mathbb{R} \). The root function \( \sqrt{x} \) is continuous on its domain, \( [0, \infty) \), and so the sum \( h(x) = \sqrt{x} + x^2 \cos x \) is continuous on its domain, \( [0, \infty) \).

47. \( f(x) = x^6 - x^3 + 3x - 5 \) is continuous on the interval \([1, 2]\), \( f(1) = -2 \), and \( f(2) = 25 \). Since \( -2 < 0 < 25 \), there is a number \( c \) in \((1, 2)\) such that \( f(c) = 0 \) by the Intermediate Value Theorem. Thus, there is a root of the equation \( x^6 - x^3 + 3x - 5 = 0 \) in the interval \((1, 2)\).
48. Let \( f(x) = 2 \sin x - 3 + 2x \). Now \( f \) is continuous on \([0, 1]\) and \( f(0) = -3 < 0 \) and \( f(1) = 2 \sin 1 - 1 \approx 0.68 > 0 \). So by the Intermediate Value Theorem there is a number \( c \) in \((0,1)\) such that \( f(c) = 0 \), that is, the equation \( 2 \sin x = 3 - 2x \) has a root in \((0,1)\).
Chapter 01-True-False Quiz

1. False. Let \( f(x) = x^2 \), \( s = -1 \), and \( t = 1 \). Then \( f(s + t) = (-1 + 1)^2 = 0^2 = 0 \), but 
\[ f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s + t). \]

2. False. Let \( f(x) = x^2 \). Then \( f(-2) = 4 = f(2) \), but \( -2 \neq 2 \).

3. False. Let \( f(x) = x^2 \). Then \( f(3x) = (3x)^2 = 9x^2 \) and \( 3f(x) = 3x^2 \). So \( f(3x) \neq 3f(x) \).

4. True. If \( x_1 < x_2 \) and \( f \) is a decreasing function, then the \( y \)-values get smaller as we move from left to right. Thus, \( f(x_1) > f(x_2) \).

5. True. See the Vertical Line Test.

6. False. Let \( f(x) = x^2 \) and \( g(x) = 2x \). Then \( (f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2 \) and 
\( (g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2 \). So \( f \circ g \neq g \circ f \).

7. False. Limit Law 2 applies only if the individual limits exist (these don’t).

8. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).


10. True. The limit doesn’t exist since \( f(x)/g(x) \) doesn’t approach any real number as \( x \) approaches 5. (The denominator approaches 0 and the numerator doesn’t.)

11. False. Consider \( \lim_{x \to 5} \frac{x(x - 5)}{x - 5} \) or \( \lim_{x \to 5} \frac{\sin(x - 5)}{x - 5} \). The first limit exists and is equal to 5. By Equation 1.4.6, we know that the latter limit exists (and it is equal to 1).

12. False. Consider \( \lim_{x \to 6} [f(x)g(x)] = \lim_{x \to 6} \left( (x - 6) \frac{1}{x - 6} \right) \). It exists (its value is 1) but \( f(6) = 0 \) and \( g(6) \) does not exist, so \( f(6)g(6) \neq 1 \).

13. True. A polynomial is continuous everywhere, so \( \lim_{x \to 6} p(x) \) exists and is equal to \( p(6) \).

14. False. Consider \( \lim_{x \to 0} [f(x) - g(x)] = \lim_{x \to 0} \left( \frac{1}{x^2} - \frac{1}{x^4} \right) \). This limit is \(-\infty \) (not 0), but each of the individual functions approaches \( \infty \).

15. True. See Figure 10 in Section 1.6.

16. False. Consider \( f(x) = \sin x \) for \( x \geq 0 \). \( \lim_{x \to -\infty} f(x) \neq \pm \infty \) and \( f \) has no horizontal asymptote.
17. False. Consider \( f(x) = \begin{cases} \frac{1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \)

18. False. For instance, the denominator of the rational function \( f(x) = \frac{x^2 - 2x}{x - 2} \) is 0 when \( x = 2 \). But if \( x \neq 2 \), then

\[
\lim_{x \to 2} \frac{x^2 - 2x}{x - 2} = \lim_{x \to 2} \frac{x(x - 2)}{x - 2} = \lim_{x \to 2} x = 2
\]

So \( \lim_{x \to 2} f(x) = 2 \) and \( x = 2 \) is not a vertical asymptote.

19. False. For example, if \( x = -3 \), then \( \sqrt{(-3)^2} = \sqrt{9} = 3 \), not \(-3\).

20. False. The function \( f \) must be \textit{continuous} in order to use the Intermediate Value Theorem. For example, let

\[
f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}
\]

There is no number \( c \in [0, 3] \) with \( f(c) = 0 \).

21. True. Use Theorem 1.5.7 with \( a = 2, b = 5 \), and \( g(x) = 4x^2 - 11 \). Note that \( f(4) = 3 \) is not needed.

22. True. Use the Intermediate Value Theorem with \( a = -1, b = 1 \), and \( N = \pi \), since \( 3 < \pi < 4 \).

23. True, by the definition of a limit with \( \epsilon = 1 \).

24. False. For example, let \( f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases} \)

Then \( f(x) > 1 \) for all \( x \), but \( \lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 + 1) = 1 \).

25. True. See Exercise 50(b) in Section 1.5.

26. False. See Exercise 50(c) in Section 1.5.