1. \[ A = \int_{x=0}^{x=4} (y_T - y_B) \, dx = \int_0^4 [(5x - x^2) - x] \, dx = \int_0^4 (4x - x^2) \, dx = [2x^2 - \frac{3}{2}x^3]_0^4 = (32 - \frac{64}{9}) - (0) = \frac{22}{9} \]

2. \[ A = \int_0^2 \left( \sqrt{x} + 2 - \frac{1}{x+1} \right) \, dx = \left[ \frac{2}{3}(x + 2)^{3/2} - \ln(x + 1) \right]_0^2 = \left[ \frac{2}{3}(4)^{3/2} - \ln 3 \right] - \left[ \frac{2}{3}(2)^{3/2} - \ln 1 \right] = \frac{16}{3} - \ln 3 - \frac{3}{2} \sqrt{2} \]

3. \[ A = \int_{y=-1}^{y=1} (x_R - x_L) \, dy = \int_{-1}^{1} [e^y - (y^2 - 2)] \, dy = \int_{-1}^{1} (e^y - y^2 + 2) \, dy \]
   \[ = [e^y - \frac{1}{2}y^2 + 2y]_{-1}^1 = (e^1 - \frac{1}{2} + 2) - (e^{-1} + \frac{1}{2} - 2) = e - \frac{1}{e} + \frac{10}{3} \]

4. \[ A = \int_0^3 [(2y - y^2) - (y^2 - 4y)] \, dy = \int_0^3 (-2y^2 + 6y) \, dy = [-\frac{2}{3}y^3 + 3y^2]_0^3 = (-18 + 27) - 0 = 9 \]

5. \[ A = \int_{-1}^{1} [e^y - (x^2 - 1)] \, dx = [e^y - \frac{1}{2}x^3 + x]_{-1}^1 \]
   \[ = (e - \frac{1}{2} + 1) - (e^{-1} + \frac{1}{2} - 1) = e - \frac{1}{e} + \frac{4}{3} \]

6. \[ A = \int_{0}^{\pi/2} (e^x - \sin x) \, dx \]
   \[ = [e^x + \cos x]_{0}^{\pi/2} \]
   \[ = (e^{\pi/2} + 0) - (1 + 1) \]
   \[ = e^{\pi/2} - 2 \]

7. The curves intersect when \( x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0, 1. \)
   \[ A = \int_0^1 (x - x^2) \, dx = \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 \]
   \[ = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \]
8. By observation, \( y = \sin x \) and \( y = \frac{2x}{\pi} \) intersect at \((0, 0)\) and \((\pi/2, 1)\) for \( x \geq 0 \).

\[
A = \int_0^{\pi/2} \left( \sin x - \frac{2x}{\pi} \right) dx = \left[ -\cos x - \frac{1}{\pi} x^2 \right]_0^{\pi/2} = \left( 0 - \frac{\pi}{4} \right) - (-1) = 1 - \frac{\pi}{4}
\]

9. \( 2y^2 = 1 - y \iff 2y^2 + y - 1 = 0 \iff (2y - 1)(y + 1) = 0 \iff y = \frac{1}{2} \) or \(-1\), so \( x = \frac{1}{2} \) or \( 2 \) and

\[
A = \int_{-1}^{1/2} [(1 - y) - 2y^2] dy = \left[ y - \frac{1}{2} y^2 - \frac{2}{3} y^3 \right]_{-1}^{1/2} = \left( \frac{1}{2} - \frac{1}{3} - \frac{1}{12} \right) - \left( -1 - \frac{1}{2} + \frac{2}{3} \right) = \frac{\sqrt{2}}{4} - \left( -\frac{8}{6} \right) = \frac{\sqrt{2}}{4} + \frac{8}{6} = \frac{\sqrt{2}}{4} + \frac{40}{24} = \frac{40 - \sqrt{2}}{24} = \frac{8}{3}
\]

10. \( 4x + x^3 = 12 \iff (x + 6)(x - 2) = 0 \iff x = -6 \) or \( x = 2 \), so \( y = -6 \) or \( y = 2 \) and

\[
A = \int_{-6}^{2} \left[ (-\frac{1}{12} y^2 + 3) - y \right] dy = \left[ -\frac{1}{12} y^2 - \frac{1}{2} y^2 + 3y \right]_{-6}^{2} = \left( -\frac{2}{2} - 2 + 6 \right) - (18 - 18 - 18) = 22 - \frac{2}{3} = \frac{64}{3}
\]
11. \[12 - x^2 = x^2 - 6 \quad \Rightarrow \quad 2x^2 = 18 \quad \Rightarrow \]
\[x^2 = 9 \quad \Rightarrow \quad x = \pm 3, \text{ so} \]
\[A = \int_{-3}^{3} [(12 - x^2) - (x^2 - 6)] \, dx \]
\[= 2 \int_{0}^{3} (18 - 2x^2) \, dx \quad \text{[by symmetry]} \]
\[= 2\left[18x - \frac{2}{3}x^3\right]_0^3 = 2[(54 - 18) - 0] \]
\[= 2(36) = 72 \]

12. \[x^2 = 4x - x^2 \quad \Rightarrow \quad 2x^2 - 4x = 0 \quad \Rightarrow \quad 2x(x - 2) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 2, \text{ so} \]
\[A = \int_{0}^{2} [(4x - x^2) - x^2] \, dx \]
\[= \int_{0}^{2} (4x - 2x^2) \, dx \]
\[= \left[2x^2 - \frac{2}{3}x^3\right]_0^2 \]
\[= 8 - \frac{16}{3} = \frac{8}{3} \]

13. \[e^x = xe^x \quad \Rightarrow \quad e^x - xe^x = 0 \quad \Rightarrow \quad e^x(1 - x) = 0 \quad \Rightarrow \quad x = 1. \]
\[A = \int_{0}^{1} (e^x - xe^x) \, dx \]
\[= \left[e^x - (xe^x - e^x)\right]_0^1 \quad \text{[use parts with } u = x \text{ and } dv = e^x \, dx] \]
\[= \left[2e^x - xe^x\right]_0^1 = (2e - e) - (2 - 0) = e - 2 \]

14. \[A = \int_{0}^{2\pi} [(2 - \cos x) - \cos x] \, dx \]
\[= \int_{0}^{2\pi} (2 - 2\cos x) \, dx \]
\[= \left[2x - 2 \sin x\right]_0^{2\pi} \]
\[= (4\pi - 0) - 0 = 4\pi \]
15. \[ 2y^2 = 4 + y^2 \iff y^2 = 4 \iff y = \pm 2, \text{ so} \]
\[
A = \int_{-2}^{2} \left[ (4 + y^2) - 2y^2 \right] dy
\]
\[
= 2 \int_{0}^{2} (4 - y^2) dy \quad \text{[by symmetry]}
\]
\[
= 2\left[4y - \frac{1}{3}y^3\right]_0^2 = 2\left(8 - \frac{8}{3}\right) = \frac{20}{3}
\]

16. \[ 1 + \sqrt{x} = 1 + \frac{1}{3}x \iff \sqrt{x} = \frac{1}{3}x \iff x = \frac{x^2}{9} \iff 9x - x^2 = 0 \iff x(9 - x) = 0 \iff x = 0 \text{ or } 9, \text{ so} \]
\[ A = \int_{0}^{9} \left[ (1 + \sqrt{x}) - (1 + \frac{1}{3}x) \right] dx = \int_{0}^{9} \left( \sqrt{x} - \frac{1}{3}x \right) dx = \left[ \frac{2}{3}x^{3/2} - \frac{1}{6}x^2 \right]_0^9 = \frac{18}{2} - \frac{27}{2} = \frac{9}{2}
\]

17. By inspection, the curves intersect at \( x = \pm \frac{3}{2} \).
\[ A = \int_{-1/2}^{1/2} \left[ \cos \pi x - (4x^2 - 1) \right] dx
\]
\[
= 2 \int_{0}^{1/2} \left( \cos \pi x - 4x^2 + 1 \right) dx \quad \text{[by symmetry]}
\]
\[
= 2\left[\frac{1}{\pi} \sin \pi x - \frac{4}{3}x^3 + x\right]_0^{1/2} = 2\left[\left(\frac{1}{\pi} - \frac{1}{6} + \frac{1}{2}\right) - 0\right]
\]
\[
= 2\left(\frac{1}{\pi} + \frac{1}{3}\right) = \frac{2}{\pi} + \frac{2}{3}
\]

18. For \( x > 0, x = x^2 - 2 \iff 0 = x^2 - x - 2 \iff 0 = (x - 2)(x + 1) \iff x = 2. \) By symmetry,
\[ A = \int_{-2}^{2} \left[ |x| - (x^2 - 2) \right] dx
\]
\[
= 2 \int_{0}^{2} \left[ x - (x^2 - 2) \right] dx = 2 \int_{0}^{2} (x - x^2 + 2) dx
\]
\[
= 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x\right]_0^2 = 2\left(2 - \frac{8}{3} + 4\right) = \frac{20}{3}
\]
19. $1/x = x \iff 1 = x^2 \iff x = \pm 1$ and $1/x = \frac{1}{4}x \iff$

$4 = x^2 \iff x = \pm 2$, so for $x > 0$,

$A = \int_0^1 \left( x - \frac{1}{4}x \right) \, dx + \int_1^2 \left( \frac{1}{x} - \frac{1}{4}x \right) \, dx$

$= \int_0^1 \left( \frac{3}{4}x \right) \, dx + \int_1^2 \left( \frac{1}{x} - \frac{1}{4}x \right) \, dx$

$= \left[ \frac{3}{8}x^2 \right]_0^1 + \left[ \ln|x| - \frac{1}{8}x^2 \right]_1^2$

$= \frac{3}{8} + \left( \ln 2 - \frac{1}{8} \right) - \left( 0 - \frac{1}{8} \right) = \ln 2$

20. $\frac{1}{3}x^2 = -x + 3 \iff x^2 + 4x - 12 = 0 \iff (x + 6)(x - 2) = 0 \iff x = -6$ or $2$ and $2x^2 = -x + 3 \iff$

$2x^2 + x - 3 = 0 \iff (2x + 3)(x - 1) = 0 \iff x = -\frac{3}{2}$ or $1$, so for $x \geq 0$,

$A = \int_0^1 \left( 2x^2 - \frac{1}{2}x^2 \right) \, dx + \int_1^2 \left[ (-x + 3) - \frac{1}{2}x^2 \right] \, dx$

$= \int_0^1 \frac{7}{4}x^2 \, dx + \int_1^2 \left( -\frac{1}{4}x^2 - x + 3 \right) \, dx$

$= \left[ \frac{7}{12}x^3 \right]_0^1 + \left[ -\frac{1}{12}x^2 - \frac{1}{2}x^2 + 3x \right]_1^2$

$= \frac{7}{12} + \left( -\frac{3}{2} - 2 + 6 \right) - \left( -\frac{1}{2} - \frac{1}{2} + 3 \right) = \frac{3}{4}$

21. $\cos x = \sin 2x = 2 \sin x \cos x \iff 2 \sin x \cos x - \cos x = 0 \iff \cos x (2 \sin x - 1) = 0 \iff$

$2 \sin x = 1$ or $\cos x = 0 \iff x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

$A = \int_0^{\pi/6} (\cos x - \sin 2x) \, dx + \int_{\pi/6}^{\pi/2} (2 \sin x - \cos x) \, dx$

$= \left[ \sin x + \frac{1}{2} \cos 2x \right]_0^{\pi/6} + \left[ -\frac{1}{2} \cos 2x - \sin x \right]_{\pi/6}^{\pi/2}$

$= \left( \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \right) - \left( 0 + \frac{1}{2} \cdot 1 \right) + \left[ -\frac{1}{2} \cdot (-1) - 1 \right] - \left( -\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \right)$

$= \frac{3}{4} - \frac{1}{2} - \frac{1}{2} + \frac{3}{4} = \frac{1}{2}$

22. $A = \int_0^1 \left[ (x^3 - 4x^2 + 3x) - (x^2 - x) \right] \, dx$\n
$\quad + \int_1^4 \left[ (x^2 - x) - (x^2 - 4x^2 + 3x) \right] \, dx$

$= \int_0^1 (x^3 - 5x^2 + 4x) \, dx + \int_1^4 (-x^2 + 5x^2 - 4x) \, dx$

$= \left[ \frac{1}{4}x^4 - \frac{5}{6}x^3 + 2x^2 \right]_0^1 + \left[ -\frac{1}{4}x^4 + \frac{5}{6}x^3 - 2x^2 \right]_1^4$

$= \left( \frac{1}{4} - \frac{5}{6} + 2 \right) - 0 + \left( -64 + \frac{250}{9} - 32 \right) - \left( -\frac{1}{4} + \frac{5}{6} - 2 \right) = \frac{71}{6}$
23. Graph \( Y_1 = \frac{2}{1+x^4} \) and \( Y_2 = x^2 \). We see that \( Y_1 > Y_2 \) on \((-1, 1)\), so the area is given by \( \int_{-1}^{1} \left( \frac{2}{1+x^4} - x^2 \right) \, dx \). Evaluate the integral with a command such as \( \text{fnInt}( Y_1 - Y_2, x, -1, 1 ) \) to get 2.80123 to five decimal places.

Another method: Graph \( f(x) = Y_1 - Y_2 \) and from the graph evaluate \( \int f(x) \, dx \) from \(-1\) to \(1\).

The curves intersect at \( x = \pm 1 \).

\[
A = \int_{-1}^{1} (e^{1-x^2} - x^4) \, dx \approx 3.66016
\]

24. The curves intersect at \( x = 0 \) and \( x = \alpha \approx 0.749363 \).

\[
A = \int_{0}^{\alpha} (\sqrt{x} - \tan^2 x) \, dx \approx 0.25142
\]

25. The curves intersect at \( x = \alpha \approx -1.911917 \), \( x = b \approx -1.223676 \), and \( x = c \approx 0.607946 \).

\[
A = \int_{a}^{b} [(x + 2 \sin^4 x) - \cos x] \, dx + \int_{b}^{c} [\cos x - (x + 2 \sin^4 x)] \, dx \\
\approx 1.70413
\]

27. As in Example 3, we approximate the distance between the two cars after ten seconds using Simpson's Rule

with \( \Delta t = 1 \, \text{s} = \frac{1}{3.6} \, \text{h} \).

\[
\text{distance}_{K} - \text{distance}_{C} = \int_{0}^{10} v_K \, dt - \int_{0}^{10} v_C \, dt = \int_{0}^{10} (v_K - v_C) \, dt \approx S_{10}
\]

\[
= \frac{1}{3.6} \left[ 0 - 0 + 4(35 - 32) + 2(59 - 51) + 4(83 - 74) + 2(98 - 86) + 4(114 - 99) + 2(128 - 110) + 4(138 - 120) + 2(150 - 130) + 4(157 - 138) + (163 - 144) \right] \\
= \frac{1}{10 \times 3.6} (391) \approx 0.0362 \, \text{km}
\]

So after 10 seconds, Kelly's car is about 36 meters ahead of Chris's.
28. We know that the area under curve $A$ between $t = 0$ and $t = x$ is $\int_0^x v_A(t) \, dt = s_A(x)$, where $v_A(t)$ is the velocity of car $A$ and $s_A$ is its displacement. Similarly, the area under curve $B$ between $t = 0$ and $t = x$ is $\int_0^x v_B(t) \, dt = s_B(x)$.

(a) After one minute, the area under curve $A$ is greater than the area under curve $B$. So car $A$ is ahead after one minute.

(b) The area of the shaded region has numerical value $s_A(1) - s_B(1)$, which is the distance by which $A$ is ahead of $B$ after 1 minute.

(c) After two minutes, car $B$ is traveling faster than car $A$ and has gained some ground, but the area under curve $A$ from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve $B$, so car $A$ is still ahead.

(d) From the graph, it appears that the area between curves $A$ and $B$ for $0 \leq t \leq 1$ (when car $A$ is going faster), which corresponds to the distance by which car $A$ is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time $x$ where the area between the curves for $1 \leq t \leq x$ (when car $B$ is going faster) is the same as the area for $0 \leq t \leq 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

29. If $x = \text{distance from left end of pool and } w = w(x) = \text{width at } x$, then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives

$$\text{Area} = \int_0^6 w \, dx \approx \frac{2}{3} \left[ 0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0 \right] = \frac{2}{3}(126.4) \approx 84 \text{ m}^2.$$

30. Let $h(x)$ denote the height of the wing at $x$ cm from the left end.

$$A \approx S_{10} = \frac{200}{3(10)} \left[ h(0) + 4h(20) + 2h(40) + \cdots + 4h(180) + h(200) \right]$$

$$= \frac{20}{3} \left[ 5.8 + 4(20.3) + 2(26.7) + 4(29.0) + 2(27.6) + 4(27.3) + 2(23.8) + 4(20.5) + 2(15.1) + 4(8.7) + 2.8 \right]$$

$$= \frac{20}{3}(618.2) \approx 4121 \text{ cm}^2$$

31. For $0 \leq t \leq 10$, $b(t) > d(t)$, so the area between the curves is given by

$$\int_0^{10} [b(t) - d(t)] \, dt = \int_0^{10} (2200e^{0.024t} - 1460e^{0.018t}) \, dt = \left[ \frac{2200}{0.024}e^{0.024t} - \frac{1460}{0.018}e^{0.018t} \right]_0^{10}$$

$$= \left( \frac{275,000}{3}e^{0.24} - \frac{730,000}{9}e^{0.18} \right) - \left( \frac{275,000}{3} - \frac{730,000}{9} \right) \approx 8868 \text{ people}$$

This area $A$ represents the increase in population over a 10-year period.
32. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is

\[ A = 2 \int_{-5}^{5} \sqrt{25 - y^2} \, dy \]

\[ = \left[ 25 \arcsin(y/5) + y \sqrt{25 - y^2} \right]_{-5}^{5} \quad \text{[substitute } y = 5 \sin \theta \text{]} \]

\[ = 25 \arcsin \frac{2}{5} + 2 \sqrt{21} + \frac{25}{2} \pi \approx 58.72 \text{ m}^2 \]

so the fraction of the total capacity in use is \( \frac{A}{\pi (5)^2} \approx \frac{58.72}{25\pi} \approx 0.748 \text{ or } 74.8\%. \)

33. Let the equation of the large circle be \( x^2 + y^2 = R^2 \). Then the equation of the small circle is \( x^2 + (y - b)^2 = r^2 \), where \( b = \sqrt{R^2 - r^2} \) is the distance between the centers of the circles. The desired area is

\[ A = \int_{-r}^{r} \left[ (b + \sqrt{r^2 - x^2}) - \sqrt{R^2 - x^2} \right] dx \]

\[ = 2 \int_{0}^{r} (b + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2}) dx \]

\[ = 2 \int_{0}^{r} b \, dx + 2 \int_{0}^{r} \sqrt{r^2 - x^2} \, dx - 2 \int_{0}^{r} \sqrt{R^2 - x^2} \, dx \]

The first integral is just \( 2br = 2r \sqrt{R^2 - r^2} \). The second integral represents the area of a quarter-circle of radius \( r \), so its value is \( \frac{1}{4} \pi r^2 \). To evaluate the other integral, note that

\[ \int \sqrt{a^2 - x^2} \, dx = \int a^2 \cos^2 \theta \, d\theta \quad \text{[} x = a \sin \theta, \, dx = a \cos \theta \, d\theta \text{]} = \left( \frac{1}{2} a^2 \right) \int (1 + \cos 2\theta) \, d\theta \]

\[ = \frac{1}{2} a^2 (\theta + \frac{1}{2} \sin 2\theta) + C = \frac{1}{2} a^2 (\theta + \sin \theta \cos \theta) + C \]

\[ = \frac{a^2}{2} \arcsin \left( \frac{x}{a} \right) + \frac{a^2}{2} \left( \frac{x}{a} \right) \sqrt{a^2 - x^2} + C = \frac{a^2}{2} \arcsin \left( \frac{x}{a} \right) + \frac{x}{2} \sqrt{a^2 - x^2} + C \]

Thus, the desired area is

\[ A = 2r \sqrt{R^2 - r^2} + 2 \left( \frac{1}{2} \pi r^2 \right) - \left[ R^2 \arcsin(x/R) + x \sqrt{R^2 - x^2} \right]_{0}^{r} \]

\[ = 2r \sqrt{R^2 - r^2} + \frac{1}{2} \pi r^2 - \left[ R^2 \arcsin(r/R) + r \sqrt{R^2 - r^2} \right] = r \sqrt{R^2 - r^2} + \frac{\pi}{2} r^2 - R^2 \arcsin(r/R) \]
34. The inequality \( x \geq 2y^2 \) describes the region that lies on, or to the right of, the parabola \( x = 2y^2 \). The inequality \( x \leq 1 - |y| \) describes the region that lies on, or to the left of, the curve \( x = 1 - |y| = \begin{cases} 1 - y & \text{if } y \geq 0 \\ 1 + y & \text{if } y < 0 \end{cases} \).

So the given region is the shaded region that lies between the curves. The graphs of \( x = 1 - y \) and \( x = 2y^2 \) intersect when \( 1 - y = 2y^2 \) \iff \( 2y^2 + y - 1 = 0 \) \iff \( (2y - 1)(y + 1) = 0 \) \iff \( y = \frac{1}{2} \) [for \( y \geq 0 \)]. By symmetry,
\[
A = 2 \int_0^{1/2} \left[(1-y) - 2y^2\right] \, dy = 2 \left[-\frac{3}{2}y^2 - \frac{1}{3}y^3 + y\right]_0^{1/2} = 2\left[-\frac{1}{12} - \frac{1}{8} + \frac{1}{2}\right] - 0 = 2\left(\frac{7}{24}\right) = \frac{7}{12}.
\]

35. We first assume that \( c > 0 \), since \( c \) can be replaced by \( -c \) in both equations without changing the graphs, and if \( c = 0 \) the curves do not enclose a region. We see from the graph that the enclosed area \( A \) lies between \( x = -c \) and \( x = c \), and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is
\[
A = 4 \int_0^c \left(c^2 - x^2\right) \, dx = 4\left[c^2x - \frac{1}{3}x^3\right]_0^c = 4\left[c^2 - \frac{1}{3}c^3\right] = 4\left(\frac{2}{3}c^3\right) = \frac{8}{3}c^3
\]
So \( A = 576 \) \iff \( \frac{2}{3}c^3 = 576 \) \iff \( c^3 = 216 \) \iff \( c = \sqrt[3]{216} = 6 \).

Note that \( c = -6 \) is another solution, since the graphs are the same.

36. We start by finding the equation of the tangent line to \( y = x^2 \) at the point \((1, 1)\):
\[
y' = 2x, \text{ so the slope of the tangent is } 2(1) = 2, \text{ and its equation is } y - 1 = 2(x - 1), \text{ or } y = 2x - 1.
\]
We would need two integrals to integrate with respect to \( x \), but only one to integrate with respect to \( y \).
\[
A = \int_1^0 \left[\frac{1}{2}(y + 1) - \sqrt{y}\right] \, dy = \left[\frac{1}{4}y^2 + \frac{1}{2}y - \frac{2}{3}y^{3/2}\right]_0^1
= \frac{1}{4} + \frac{1}{2} - \frac{2}{3} = \frac{1}{12}
\]
37. By the symmetry of the problem, we consider only the first quadrant, where \( y = x^2 \) \iff \( x = \sqrt{y} \). We are looking for a number \( b \) such that
\[
\int_0^b \sqrt{y} \, dy = \int_b^4 \sqrt{y} \, dy \quad \Rightarrow \quad \frac{2}{3} \left[y^{3/2}\right]_0^b = \frac{2}{3} \left[y^{3/2}\right]_b^4 \quad \Rightarrow \\
\frac{2}{3}b^{3/2} = 4^{3/2} - b^{3/2} \quad \Rightarrow \quad 2b^{3/2} = 8 \quad \Rightarrow \quad b^{3/2} = 4 \quad \Rightarrow \quad b = 4^{2/3} \approx 2.52.
\]
38. (a) We want to choose \( a \) so that
\[
\int_a^\infty \frac{1}{x^2} \, dx = \int_1^a \frac{1}{x^2} \, dx \quad \Rightarrow \quad \left[ -\frac{1}{x} \right]_1^a = \left[ -\frac{1}{x} \right]_a^1 \quad \Rightarrow \quad -\frac{1}{a} + 1 = -\frac{1}{1} + \frac{1}{a} \quad \Rightarrow \quad \frac{5}{4} = \frac{2}{a} \quad \Rightarrow \quad a = \frac{8}{5}
\]

(b) The area under the curve \( y = \frac{1}{x^2} \) from \( x = 1 \) to \( x = 4 \) is \( \frac{7}{4} \) [take \( a = 4 \) in the first integral in part (a)]. Now the line \( y = b \) must intersect the curve \( x = 1/\sqrt{y} \) and not the line \( x = 4 \), since the area under the line \( y = 1/4^2 \) from \( x = 1 \) to \( x = 4 \) is only \( \frac{2}{25} \), which is less than half of \( \frac{7}{4} \). We want to choose \( b \) so that the upper area in the diagram is half of the total area under the curve \( y = 1/x^2 \) from \( x = 1 \) to \( x = 4 \). This implies that
\[
\int_1^{\frac{1}{\sqrt{b}}} (1/\sqrt{y} - 1) \, dy = \frac{1}{2} \cdot \frac{2}{4} \quad \Rightarrow \quad [2\sqrt{y} - y]_b^1 = \frac{2}{3} \quad \Rightarrow \quad 1 - 2\sqrt{b} + b = \frac{2}{3} \quad \Rightarrow \quad b - 2\sqrt{b} + \frac{5}{3} = 0.
\]
Letting \( c = \sqrt{b} \), we get \( c^2 - 2c + \frac{5}{3} = 0 \quad \Rightarrow \quad 8c^2 - 16c + 5 = 0 \). Thus, \( c = \frac{16 \pm \sqrt{256 - 4 \cdot 16}}{16} = 1 \pm \frac{\sqrt{15}}{4} \). But \( c = \sqrt{b} < 1 \quad \Rightarrow \quad c = 1 - \frac{\sqrt{15}}{4} \quad \Rightarrow \quad b = c^2 = 1 + \frac{3}{2} - \frac{\sqrt{15}}{2} = \frac{1}{4} (11 - 4\sqrt{15}) \approx 0.1503.

39. The area under the graph of \( f \) from 0 to \( t \) is equal to \( \int_0^t f(x) \, dx \), so the requirement is that \( \int_0^t f(x) \, dx = t^2 \) for all \( t \). We differentiate both sides of this equation with respect to \( t \) (with the help of FTC1) to get \( f(t) = 3t^2 \). This function is positive and continuous, as required.

40. It appears from the diagram that the curves \( y = \cos x \) and \( y = \cos(x - c) \) intersect halfway between 0 and \( c \), namely, when \( x = c/2 \). We can verify that this is indeed true by noting that \( \cos(c/2 - c) = \cos(-c/2) = \cos(c/2) \). The point where \( \cos(x - c) \) crosses the \( x \)-axis is \( x = \frac{\pi}{2} + c \). So we require that
\[
\int_0^{\pi/2} \cos x - \cos(x - c) \, dx = -\int_{\pi/2 + c}^\pi \cos(x - c) \, dx \quad \text{[the negative sign on the RHS is needed since the second area is beneath the x-axis]}
\]
\[
\Rightarrow \quad [\sin x - \sin(x - c)]_0^{\pi/2} = -[\sin(x - c)]_{\pi/2 + c}^\pi \quad \Rightarrow \quad [\sin(c/2) - \sin(-c/2)] - [-\sin(-c)] = -\sin(\pi/2) + \sin[(\pi/2) + c] \quad \Rightarrow \quad 2\sin(c/2) - \sin c = -\sin c + 1.
\]
[Here we have used the oddness of the sine function, and the fact that \( \sin(\pi - c) = \sin c \). So \( 2\sin(c/2) = 1 \) \( \Rightarrow \) \( \sin(c/2) = \frac{1}{2} \) \( \Rightarrow \) \( c/2 = \frac{\pi}{6} \) \( \Rightarrow \) \( c = \frac{\pi}{3} \).]
41. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2 + 1) = mx \implies \ x = x(mx^2 + m) \implies x(mx^2 + m) - x = 0 \implies x(mx^2 + m - 1) = 0 \implies x = 0$ or $mx^2 + m - 1 = 0$ \implies

$x = 0$ or $x^2 = \frac{1 - m}{m} \implies x = 0$ or $x = \pm \sqrt{\frac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \implies 1/m > 1 \implies 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2 + 1)$ at the origin is $y'(0) = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since $mx$ and $x/(x^2 + 1)$ are both odd functions, the total area is twice the area between the curves on the interval $[0, \sqrt{1/m - 1}]$. So the total area enclosed is

$$2 \int_0^{\sqrt{1/m - 1}} \left( \frac{x}{x^2 + 1} - mx \right) dx = 2 \left[ \frac{1}{2} \ln(x^2 + 1) - \frac{1}{2} mx^2 \right]_{0}^{\sqrt{1/m - 1}} = \left[ \ln(1/m - 1 + 1) - m(1/m - 1) \right] - (\ln 1 - 0)
$$

$$= \ln(1/m) - 1 + m = m - \ln m - 1$$
1. A cross-section is a disk with radius $2 - \frac{1}{2}x$, so its area is $A(x) = \pi \left(2 - \frac{1}{2}x\right)^2$.

\[
V = \int_1^2 A(x) \, dx = \int_1^2 \pi \left(2 - \frac{1}{2}x\right)^2 \, dx \\
= \pi \int_1^2 \left(4 - 2x + \frac{1}{4}x^2\right) \, dx \\
= \pi \left[4x - x^2 + \frac{1}{12}x^3\right]_1^2 \\
= \pi \left[(8 - 4 + \frac{8}{12}) - (4 - 1 + \frac{1}{12})\right] \\
= \pi \left(1 + \frac{7}{12}\right) = \frac{39}{12} \pi
\]

2. A cross-section is a disk with radius $1 - x^2$, so its area is $A(x) = \pi (1 - x^2)^2$.

\[
V = \int_{-1}^1 A(x) \, dx = \int_{-1}^1 \pi (1 - x^2)^2 \, dx \\
= 2\pi \int_0^1 (1 - 2x^2 + x^4) \, dx = 2\pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 \\
= 2\pi \left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\pi \left(\frac{4}{15}\right) = \frac{16}{15} \pi
\]

3. A cross-section is a disk with radius $2\sqrt{y}$, so its area is $A(y) = \pi \left(2\sqrt{y}\right)^2$.

\[
V = \int_0^9 A(y) \, dy = \int_0^9 \pi \left(2\sqrt{y}\right)^2 \, dy = 4\pi \int_0^9 y \, dy \\
= 4\pi \left[\frac{1}{2}y^2\right]_0^9 = 2\pi (81) = 162\pi
\]

4. A cross-section is a disk with radius $e^y$ [since $y = \ln x$], so its area is $A(y) = \pi (e^y)^2$.

\[
V = \int_1^2 \pi (e^y)^2 \, dy = \pi \int_1^2 e^{2y} \, dy = \pi \left[\frac{1}{2}e^{2y}\right]_1^2 = \frac{\pi}{2} \left(e^4 - e^2\right)
\]
5. A cross-section is a washer (annulus) with inner radius $x^2$ and outer radius $x$, so its area is
\[ A(x) = \pi (x^2) - \pi (x^3)^2 = \pi (x^2 - x^6). \]
\[ V = \int_0^1 A(x) \, dx = \int_0^1 \pi (x^2 - x^6) \, dx \]
\[ = \pi \left[ \frac{1}{3} x^2 - \frac{1}{7} x^7 \right]_0^1 = \pi \left( \frac{1}{3} - \frac{1}{7} \right) = \frac{4}{21} \pi \]

6. A cross-section is a washer with inner radius $\frac{1}{2} x^2$ and outer radius $5 - x^2$, so its area is
\[ A(x) = \pi \left( 5 - x^2 \right)^2 - \pi \left( \frac{1}{2} x^2 \right)^2 \]
\[ = \pi \left( 25 - 10 x^2 + x^4 - \frac{1}{4} x^4 \right). \]
\[ V = \int_{-2}^2 A(x) \, dx = \int_{-2}^2 \pi \left( 25 - 10 x^2 + \frac{15}{16} x^4 \right) \, dx \]
\[ = 2\pi \int_0^2 \left( 25 - 10 x^2 + \frac{15}{16} x^4 \right) \, dx \]
\[ = 2\pi \left[ 25x - \frac{5}{3} x^3 + \frac{9}{16} x^5 \right]_0^2 = 2\pi (50 - \frac{80}{9} + 6) = \frac{126\pi}{2} \]

7. A cross-section is a washer with inner radius $y^2$ and outer radius $2y$, so its area is
\[ A(y) = \pi \left( 2y^2 \right)^2 - \pi \left( y^2 \right)^2 = \pi (4y^2 - y^4). \]
\[ V = \int_0^2 A(y) \, dy = \pi \int_0^2 (4y^2 - y^4) \, dy \]
\[ = \pi \left[ \frac{4}{3} y^3 - \frac{1}{5} y^5 \right]_0^2 = \pi \left( \frac{32}{3} - \frac{32}{5} \right) = \frac{64}{15} \pi \]

8. $y = x^{2/3} \Leftrightarrow x = y^{3/2}$, so a cross-section is a washer with inner radius $y^{3/2}$ and outer radius 1, and its area is
\[ A(y) = \pi (1)^2 - \pi \left( y^{3/2} \right)^2 = \pi (1 - y^3). \]
\[ V = \int_0^1 A(y) \, dy = \pi \int_0^1 (1 - y^3) \, dy \]
\[ = \pi \left[ y - \frac{1}{4} y^4 \right]_0^1 = \frac{3}{4} \pi \]
9. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is

$$A(x) = \pi (1 - x)^2 - \pi \left(1 - \sqrt{x}\right)^2$$
$$= \pi \left[ (1 - 2x + x^2) - \left(1 - 2\sqrt{x} + x\right) \right]$$
$$= \pi \left( -3x + x^2 + 2\sqrt{x} \right).$$

$$V = \int_0^1 A(x) \, dx = \pi \int_0^1 \left(-3x + x^2 + 2\sqrt{x}\right) \, dx$$
$$= \pi \left[ -\frac{3}{3} x^2 + \frac{1}{3} x^3 + \frac{4}{3} x^{3/2} \right]_0^1 = \pi \left(-\frac{3}{3} + \frac{1}{3} + \frac{4}{3} \right) = \frac{\pi}{3}.$$

10. $$V = \int_1^3 \pi \left\{ \left[ \frac{1}{x} - (-1) \right]^2 - [0 - (-1)]^2 \right\} \, dx$$

$$= \pi \int_1^3 \left[ \left( \frac{1}{x} + 1 \right)^2 - 1^2 \right] \, dx$$

$$= \pi \int_1^3 \left( \frac{1}{x^2} + \frac{2}{x} \right) \, dx = \pi \left[ \frac{1}{x} + 2 \ln x \right]_1^3$$

$$= \pi \left[ (-\frac{1}{3} + 2 \ln 3) - (-1 + 0) \right]$$

$$= \pi \left( 2 \ln 3 + \frac{2}{3} \right) = 2\pi \left( \ln 3 + \frac{1}{3} \right).$$

11. $y = x^2 \Rightarrow x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$$V = \int_0^1 \pi \left\{ \left[ \sqrt{y} - (-1) \right]^2 - [y^2 - (-1)]^2 \right\} \, dy = \pi \int_0^1 \left[ \left( \sqrt{y} + 1 \right)^2 - (y^2 + 1)^2 \right] \, dy$$

$$= \pi \int_0^1 \left( y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1 \right) \, dy = \pi \int_0^1 \left( y + 2\sqrt{y} - y^4 - 2y^2 \right) \, dy$$

$$= \pi \left[ \frac{1}{2} y^2 + \frac{4}{3} y^{3/2} - \frac{1}{5} y^5 - \frac{2}{3} y^3 \right]_0^1 = \pi \left( \frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3} \right) = \frac{29}{30} \pi.$$
12. \( y = \sqrt{2x} \Rightarrow x = y^2 \), so the outer radius is \( 2 - y^2 \).

\[
V = \int_0^1 \pi \left[ (2 - y^2)^2 - (2 - y)^2 \right] \, dy
\]
\[
= \pi \int_0^1 \left[ (4 - 4y^2 + y^4) - (4 - 4y + y^2) \right] \, dy
\]
\[
= \pi \int_0^1 (y^4 - 5y^2 + 4y) \, dy
\]
\[
= \pi \left[ \frac{1}{5}y^5 - \frac{5}{3}y^3 + 2y^2 \right]_0^1
\]
\[
= \pi \left( \frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8}{15} \pi
\]

13. A cross-section is a disk with radius \( \frac{1}{x} \), so its area is \( A(x) = \pi \left( \frac{1}{x} \right)^2 \).

\[
V = \int_1^2 A(x) \, dx = \int_1^2 \pi \left( \frac{1}{x} \right)^2 \, dx
\]
\[
= \pi \int_1^2 \frac{1}{x^2} \, dx = \pi \left[ -\frac{1}{x} \right]_1^2
\]
\[
= \pi \left[ -\frac{1}{2} - (-1) \right] = \frac{\pi}{2}
\]

14. A cross-section is a disk with radius \( 2y - y^2 \), so its area is

\[
A(y) = \pi (2y - y^2)^2.
\]

\[
V = \int_0^2 A(y) \, dy = \int_0^2 \pi (2y - y^2)^2 \, dy
\]
\[
= \pi \int_0^2 (4y^2 - 4y^2 + y^4) \, dy
\]
\[
= \pi \left[ \frac{4}{2}y^2 - y^4 + \frac{1}{5}y^5 \right]_0^1
\]
\[
= \pi \left[ \left( \frac{22}{5} - 16 + \frac{22}{5} \right) - 0 \right] = \frac{16}{15} \pi
\]
15. The curves $x - y = 1$ and $y = x^2 - 4x + 3$ intersect when

$$x - 1 = x^2 - 4x + 3 \iff 0 = x^2 - 5x + 4 \iff 0 = (x - 1)(x - 4) \iff x = 1 \text{ or } 4.$$ A cross-section is a washer with inner radius $3 - (x - 1)$ and outer radius $3 - (x^2 - 4x + 3)$, so its area is

$$A(x) = \pi [3 - (x^2 - 4x + 3)]^2 - \pi [3 - (x - 1)]^2.$$  

$$V = \int_1^4 A(x) \, dx = \pi \int_1^4 \left\{ [3 - (x^2 - 4x + 3)]^2 - [3 - (x - 1)]^2 \right\} \, dx$$

$$= \pi \int_1^4 [(4x - x^3)^2 - (4 - x)^2] \, dx = \pi \int_1^4 (16x^2 - 8x^3 + x^4 - 16 + 8x - x^2) \, dx$$

$$= \pi \int_1^4 (x^4 - 8x^2 + 15x^2 + 8x - 16) \, dx = \pi \left[ \frac{1}{5} x^5 - 2x^4 + 5x^2 + 4x^2 - 16x \right]_1^4$$

$$= \pi \left[ \left( \frac{1024}{5} - 512 + 320 + 64 - 64 \right) - \left( \frac{1}{5} - 2 + 5 + 4 - 16 \right) \right] = \pi \left( \frac{1022}{5} - 183 \right) = \frac{188}{5} \pi$$

16. $V = \int_{-1}^{1} \pi (1 - y^2)^2 \, dy = 2 \int_{0}^{1} \pi (1 - y^2)^2 \, dy$

$$= 2\pi \int_{0}^{1} (1 - 2y^2 + y^4) \, dy$$

$$= 2\pi \left[ y - \frac{2}{3} y^3 + \frac{1}{5} y^5 \right]_0^1$$

$$= 2\pi \cdot \frac{8}{15} = \frac{16}{15} \pi$$

17. $y = \sqrt{x}$ $\Rightarrow$ $x = y^2$ and $y = x^2$ $\Rightarrow$ $x = \sqrt[4]{y}$. A cross-section is a washer with inner radius $1 - \sqrt[4]{y}$ and outer radius $1 - y^2$, so its area is

$$A(y) = \pi (1 - y^2)^2 - \pi (1 - \sqrt[4]{y})^2.$$  

$$V = \int_{0}^{1} A(y) \, dy = \int_{0}^{1} \left[ \pi (1 - y^2)^2 - \pi (1 - \sqrt[4]{y})^2 \right] \, dy$$

$$= \pi \int_{0}^{1} \left[ (1 - 2y^2 + y^4) - (1 - 2y^{1/2} + y^{1/2}) \right] \, dy$$

$$= \pi \int_{0}^{1} (-2y^2 + y^4 + 2y^{1/2} - y^{1/2}) \, dy = \pi \left[ -\frac{2}{3} y^3 + \frac{1}{5} y^5 + \frac{2}{3} y^{1/3} - \frac{3}{5} y^{1/3} \right]_0^1 = \pi \left( -\frac{2}{3} + \frac{1}{5} + \frac{2}{3} - \frac{3}{5} \right) = \frac{12\pi}{30}$$
18. A cross-section is a washer with inner radius \(1 - \sqrt{x}\) and outer radius \(1 - x^2\), so its area is
\[ A(x) = \pi(1 - x^2)^2 - \pi(1 - \sqrt{x})^2. \]
\[ V = \int_0^1 A(x) \, dx = \int_0^1 \left[ \pi(1 - x^2)^2 - \pi(1 - \sqrt{x})^2 \right] \, dx = \pi \int_0^1 \left[ (1 - 2x^2 + x^4) - (1 - 2x^{1/2} + x) \right] \, dx \]
\[ = \pi \int_0^1 (-2x^2 + x^6 + 2x^{1/2} - x) \, dx = \pi \left[ \frac{1}{2}x^4 + \frac{1}{7}x^7 + \frac{3}{32}x^{3/2} - \frac{1}{2}x \right]_0^1 = \pi \left( \frac{1}{2} + \frac{1}{7} + \frac{3}{32} - \frac{1}{2} \right) = \frac{10\pi}{21}. \]

19. (a) About the \(x\)-axis:
\[ V = \int_{-1}^1 \pi(e^{-x^2})^2 \, dx = 2\pi \int_0^1 e^{-2x^2} \, dx \quad \text{[by symmetry]} \]
\[ \approx 3.75825 \]

(b) About \(y = -1\):
\[ V = \int_{-1}^1 \pi \left( [e^{-x^2} - (-1)^2] - [0 - (-1)^2] \right) \, dx \]
\[ = 2\pi \int_0^1 [(e^{-x^2} + 1)^2 - 1] \, dx = 2\pi \int_0^1 (e^{-2x^2} + 2e^{-x^2}) \, dx \]
\[ \approx 13.14312 \]

20. (a) About the \(x\)-axis:
\[ V = \int_{-\pi/2}^{\pi/2} \pi(\cos^2 x)^2 \, dx = 2\pi \int_0^{\pi/2} \cos^4 x \, dx \quad \text{[by symmetry]} \]
\[ \approx 3.70110 \]

(b) About \(y = 1\):
\[ V = \int_{-\pi/2}^{\pi/2} \pi \left[ (1 - 0)^2 - (1 - \cos^2 x)^2 \right] \, dx \]
\[ = 2\pi \int_0^{\pi/2} \left[ 1 - (1 - 2\cos^2 x + \cos^4 x) \right] \, dx \]
\[ = 2\pi \int_0^{\pi/2} (2\cos^2 x - \cos^4 x) \, dx \approx 6.16850 \]
21. (a) About $y = 2$:

$$x^2 + 4y^2 = 4 \implies 4y^2 = 4 - x^2 \implies y^2 = 1 - \frac{x^2}{4} \implies$$

$$y = \pm \sqrt{1 - \frac{x^2}{4}}$$

$$V = \int_{-2}^{2} \pi \left\{ \left[ 2 - \left( \sqrt{1 - \frac{x^2}{4}} \right) \right]^2 - \left( 2 - \sqrt{1 - \frac{x^2}{4}} \right)^2 \right\} dx$$

$$= 2\pi \int_{0}^{2} 8\sqrt{1 - \frac{x^2}{4}} dx \approx 78.95684$$

(b) About $x = 2$:

$$x^2 + 4y^2 = 4 \implies x^2 = 4 - 4y^2 \implies x = \pm \sqrt{4 - 4y^2}$$

$$V = \int_{-1}^{1} \pi \left\{ \left[ 2 - \left( \sqrt{4 - 4y^2} \right) \right]^2 - \left( 2 - \sqrt{4 - 4y^2} \right)^2 \right\} dy$$

$$= 2\pi \int_{0}^{1} 8\sqrt{4 - 4y^2} dy \approx 78.95684$$

[Notice that this is the same approximation as in part (a). This can be explained by Pappus’s Theorem in Section 7.6.]

22. (a) About the $x$-axis:

$$y = x^2 \text{ and } x^2 + y^2 = 1 \implies x^2 + x^4 = 1 \implies x^4 + x^2 - 1 = 0 \implies$$

$$x = \frac{-1 + \sqrt{5}}{2} \approx 0.618 \implies x = \pm \alpha = \pm \sqrt{\frac{-1 + \sqrt{5}}{2}} \approx \pm 0.786$$

$$V = \int_{-\alpha}^{\alpha} \pi \left( \sqrt{1 - x^2} \right)^2 - (x^2)^2 \right\} dx = 2\pi \int_{0}^{\alpha} (1 - x^2 - x^4) dx$$

$$= 3.54459$$

(b) About the $y$-axis:

$$V = \int_{0}^{\alpha} \pi (\sqrt{y})^2 dy + \int_{\alpha}^{1} \pi \left( \sqrt{1 - y^2} \right)^2 dy$$

$$= \pi \int_{0}^{\alpha} y dy + \pi \int_{\alpha}^{1} (1 - y^2) dy \approx 0.99998$$

23. 

$y = x^2$ and $y = \ln(x + 1)$ intersect at $x = 0$ and at $x = \alpha \approx 0.747$.

$$V = \pi \int_{0}^{\alpha} \left\{ [\ln(x + 1)]^2 - (x^2)^2 \right\} dx \approx 0.132$$
24. \[ y = e^{\sqrt{x}} + e^{-3x} \] \[ y = 3 \sin(x^2) \text{ and } y = e^{x/2} + e^{-2x} \] intersect at 
\[ x = a \approx 0.772 \text{ and at } x = b \approx 1.524. \]
\[ V = \pi \int_a^b \left\{ [3 \sin(x^2)]^2 - (e^{x/2} + e^{-2x})^2 \right\} \, dx \approx 7.519 \]

25. \[ V = \pi \int_0^\pi \left\{ [\sin^2 x - (-1)]^2 - [0 - (-1)]^2 \right\} \, dx \]
\[ \text{CAS: } \frac{11}{8} \pi^2 \]

26. \[ V = \pi \int_0^2 \left[ (3 - x)^2 - (3 - xe^{1-x/2})^2 \right] \, dx \]
\[ \text{CAS: } \pi (-2e^2 + 24e - \frac{142}{3}) \]

27. (a) \[ \pi \int_0^{\pi/2} \cos^2 x \, dx \] describes the volume of the solid obtained by rotating the region 
\[ \mathcal{R} = \{ (x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x \} \] of the \( xy \)-plane about the \( x \)-axis.

(b) \[ \pi \int_0^1 (y^4 - y^8) \, dy = \pi \int_0^1 [(y^2)^2 - (y^4)^2] \, dy \] describes the volume of the solid obtained by rotating the region 
\[ \mathcal{R} = \{ (x, y) \mid 0 \leq y \leq 1, y^4 \leq x \leq y^2 \} \] of the \( xy \)-plane about the \( y \)-axis.
28. (a) \( \pi \int_2^5 y \, dy = \pi \int_2^5 \left( \sqrt{y} \right)^2 \, dy \) describes the volume of the solid obtained by rotating the region
\[
\mathcal{R} = \left\{ (x, y) \mid 2 \leq y \leq 5, 0 \leq x \leq \sqrt{y} \right\}
\]
of the \( xy \)-plane about the \( y \)-axis.

(b) \( \pi \int_0^{\pi/2} \left[ (1 + \cos x)^2 - 1^2 \right] \, dx \) describes the volume of the solid obtained by rotating the region
\[
\mathcal{R} = \left\{ (x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 1 \leq y \leq 1 + \cos x \right\}
\]
of the \( xy \)-plane about the \( x \)-axis.

Or: The solid could be obtained by rotating the region \( \mathcal{R}' = \left\{ (x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x \right\} \) about the line \( y = -1 \).

29. There are 10 subintervals over the 15-cm length, so we’ll use \( n = 10/2 = 5 \) for the Midpoint Rule.
\[
V = \int_0^1 A(x) \, dx \approx M_5 = \frac{18 - 9}{5} \left[ A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(13.5) \right]
\]
\[
= 3(18 + 79 + 106 + 128 + 39) = 3 \cdot 370 = 1110 \text{ cm}^3
\]

30. \( V = \int_0^1 A(x) \, dx \approx M_5 = \frac{10 - 9}{5} \left[ A(1) + A(3) + A(5) + A(7) + A(9) \right]
\]
\[
= 2(0.65 + 0.61 + 0.59 + 0.55 + 0.50) = 2(2.90) = 5.80 \text{ m}^3
\]

31. We’ll form a right circular cone with height \( h \) and base radius \( r \) by revolving the line \( y = \frac{r}{h} x \) about the \( x \)-axis.
\[
V = \pi \int_0^h \left( \frac{r}{h} x \right)^2 \, dx = \pi \int_0^h \frac{r^2}{h^2} x^2 \, dx = \pi \frac{r^2}{h^2} \left[ \frac{1}{3} x^3 \right]_0^h
\]
\[
= \pi \frac{r^2}{h^2} \left( \frac{1}{3} h^3 \right) = \frac{1}{3} \pi r^2 h
\]

Another solution: Revolve \( x = \frac{r}{h} y + r \) about the \( y \)-axis.
\[
V = \pi \int_0^h \left( \frac{r}{h} y + r \right)^2 \, dy = \pi \int_0^h \left[ \frac{r^2}{h^2} y^2 + \frac{2r^2}{h} y + r^2 \right] \, dy
\]
\[
= \pi \left[ \frac{r^2}{3h^2} y^3 + \frac{r^2}{h} y^2 + r^2 y \right]_0^h = \pi \left( \frac{1}{3} r^2 h - r^2 h + r^2 h \right) = \frac{1}{3} \pi r^3 h
\]

* Or use substitution with \( u = r - \frac{r}{h} y \) and \( du = -\frac{r}{h} \, dy \) to get
\[
\pi \int_0^r u^2 \left( -\frac{h}{r} \, du \right) = -\pi \frac{h}{r} \left[ \frac{1}{3} u^3 \right]_0^r = -\pi \frac{h}{r} \left( -\frac{1}{3} r^3 \right) = \frac{1}{3} \pi r^2 h.
\]
32. \[ V = \pi \int_0^h \left( R - \frac{R - r}{h} y \right)^2 dy \]
\[ = \pi \int_0^h \left[ R^2 - \frac{2R(R - r)}{h} y + \left( \frac{R - r}{h} y \right)^2 \right] dy \]
\[ = \pi \left[ R^2 y - \frac{R(R - r)}{h} y^2 + \frac{1}{3} \left( \frac{R - r}{h} y \right)^3 \right]_0^h \]
\[ = \pi \left[ R^2 h - R(R - r)h + \frac{1}{3} (R - r)^2 h \right] \]
\[ = \frac{1}{3} \pi h \left[ 3Rh + (R^2 - 2Rh + r^2) \right] = \frac{1}{3} \pi h (R^2 + Rr + r^2) \]

Another solution: \( \frac{H}{R} = \frac{H - h}{\tau} \) by similar triangles. Therefore, \( Hr = H(R - h) \Rightarrow hR = H(R - r) \Rightarrow \]
\[ H = \frac{hR}{R - r} \]. Now

\[ V = \frac{1}{3} \pi R^2 H - \frac{1}{3} \pi r^2 (H - h) \quad \text{[by Exercise 31]} \]
\[ = \frac{1}{3} \pi R^2 \frac{hR}{R - r} - \frac{1}{3} \pi r^2 \frac{rh}{(R - r)} \left[ H - h = \frac{rhR}{R(R - r)} \right] \]
\[ = \frac{1}{3} \pi h \frac{R^2 - r^2}{R - r} = \frac{1}{3} \pi h (R^2 + Rr + r^2) \]
\[ = \frac{1}{3} \left[ \pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)} \right] h = \frac{1}{3} (A_1 + A_2 + \sqrt{A_1 A_2}) h \]

where \( A_1 \) and \( A_2 \) are the areas of the bases of the frustum. (See Exercise 34 for a related result.)

33. \( x^2 + y^2 = r^2 \Rightarrow x^2 = r^2 - y^2 \)

\[ V = \pi \int_{r-h}^{r} (r^2 - y^2) dy = \pi \left[ y^2 - \frac{y^3}{3} \right]_{r-h}^{r} = \pi \left\{ \left[ r^2 - \frac{r^3}{3} \right] - \left[ r^2(r - h) - \frac{(r - h)^2}{3} \right] \right\} \]
\[ = \pi \left\{ \frac{2}{3} r^3 - \frac{7}{3} (r - h)^2 \right\} \]
\[ = \frac{1}{2} \pi \left\{ 2r^2 - (r - h) \left[ 3r^2 - (r - h)^2 \right] \right\} \]
\[ = \frac{1}{2} \pi \left\{ 2r^2 - (r - h) \left[ 2r^2 + 2rh + h^2 \right] \right\} \]
\[ = \frac{1}{2} \pi \left( 2r^2 - 2r^2 - 2r^2 h + rh^2 + 2r^2 h + 2rh^2 - h^2 \right) \]
\[ = \frac{1}{2} \pi (3rh^2 - h^3) = \frac{1}{2} \pi h^2 (3r - h), \text{ or, equivalently, } \pi h^2 \left( r - \frac{h}{3} \right) \]
34. An equation of the line is \( x = \frac{\Delta x}{\Delta y} y + (x\text{-intercept}) = \frac{a/2 - b/2}{h - 0} y + \frac{b}{2} = \frac{a - b}{2h} y + \frac{b}{2} \).

\[
V = \int_0^h A(y) \, dy = \int_0^h (2x)^2 \, dy \\
= \int_0^h \left[ 2 \left( \frac{a - b}{2h} y + \frac{b}{2} \right) \right]^2 \, dy = \int_0^h \left[ \frac{a - b}{h} y + b \right]^2 \, dy \\
= \int_0^h \left[ \frac{(a - b)^2}{h^2} y^2 + 2\frac{b(a - b)}{h} y + b^2 \right] \, dy \\
= \left[ \frac{(a - b)^2}{3h^2} y^3 + \frac{b(a - b)}{h} \frac{y^2}{2} + b^2 y \right]_0^h \\
= \frac{1}{3} (a - b)^2 h + b(a - b)h + b^2 h = \frac{1}{2} (a^2 - 2ab + b^2 + 3ab) h \\
= \frac{1}{2} (a^2 + ab + b^2) h
\]

[Note that this can be written as \( \frac{1}{2} (A_1 + A_2 + \sqrt{A_1 A_2}) h \), as in Exercise 32.]

If \( a = b \), we get a rectangular solid with volume \( b^2 h \). If \( a = 0 \), we get a square pyramid with volume \( \frac{1}{3} b^2 h \).

35. For a cross-section at height \( y \), we see from similar triangles that \( \frac{\alpha/2}{b/2} = \frac{h - y}{h} \), so \( \alpha = b \left( 1 - \frac{y}{h} \right) \).

Similarly, for cross-sections having \( 2b \) as their base and \( \beta \) replacing \( \alpha, \beta = 2b \left( 1 - \frac{y}{h} \right) \). So

\[
V = \int_0^h A(y) \, dy = \int_0^h \left[ b \left( 1 - \frac{y}{h} \right) \right] \left[ 2b \left( 1 - \frac{y}{h} \right) \right] \, dy \\
= \int_0^h 2b^2 \left( 1 - \frac{y}{h} \right)^2 \, dy = 2b^2 \int_0^h \left( 1 - \frac{2y}{h} + \frac{y^2}{h^2} \right) \, dy \\
= 2b^2 \left[ y - \frac{y^2}{h} + \frac{y^3}{3h^2} \right]_0^h = 2b^2 [h - h + \frac{1}{3} h] \\
= \frac{2}{3} b^2 h \quad [ = \frac{1}{3} Bh \text{ where } B \text{ is the area of the base, as with any pyramid}]
\]
36. Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height \( y \), so \( \frac{a}{b} = \frac{\alpha}{\beta} \implies \alpha = \frac{a\beta}{b} \). Also by similar triangles, \( \frac{b}{h} = \frac{\beta}{(h - y)} \implies \beta = b(h - y)/h \). These two equations imply that \( \alpha = a(1 - y/h) \), and since the cross-section is an equilateral triangle, it has area

\[
A(y) = \frac{1}{2} \cdot \alpha \cdot \frac{\sqrt{3}}{2} = \frac{a^2}{4} \sqrt{3} \left(1 - \frac{y}{h}\right)^2, \text{ so}
\]

\[
V = \int_0^h A(y) \, dy = \frac{a^2}{4} \sqrt{3} \int_0^h \left(1 - \frac{y}{h}\right)^2 \, dy
\]

\[
= \frac{a^2 \sqrt{3}}{4} \left[ \frac{h}{3} \left(1 - \frac{y}{h}\right)^3 \right]_0^h = -\frac{\sqrt{3}}{12} a^2 h(-1) = \frac{\sqrt{3}}{12} a^2 h
\]

37. A cross-section at height \( z \) is a triangle similar to the base, so we'll multiply the legs of the base triangle, 3 and 4, by a proportionality factor of \((5 - z)/5\). Thus, the triangle at height \( z \) has area

\[
A(z) = \frac{1}{2} \cdot 3 \left(\frac{5 - z}{5}\right) \cdot 4 \left(\frac{5 - z}{5}\right) = 6 \left(1 - \frac{z}{5}\right)^2, \text{ so}
\]

\[
V = \int_0^5 A(z) \, dz = 6 \int_0^5 \left(1 - \frac{z}{5}\right)^2 \, dz = 6 \int_1^5 u^2 (-5 \, du) \quad \left[ u = 1 - \frac{z}{5}, \quad \frac{du}{-5} = \frac{dz}{1} \right]
\]

\[
= -30 \left[\frac{1}{3} u^3\right]_1^5 = -30 \left(-\frac{1}{3}\right) = 10 \, \text{cm}^2
\]

38. A cross-section is shaded in the diagram.

\[
A(x) = (2y)^2 = (2 \sqrt{r^2 - x^2})^2, \text{ so}
\]

\[
V = \int_{-r}^r A(x) \, dx = 2 \int_0^r 4(r^2 - x^2) \, dx
\]

\[
= 8[r^2 x - \frac{1}{3} x^3]_0^r = 8 \left(\frac{2}{3} r^3\right) = \frac{16}{3} r^3
\]

39. If \( l \) is a leg of the isosceles right triangle and \( 2y \) is the hypotenuse,

then \( l^2 + l^2 = (2y)^2 \implies 2l^2 = 4y^2 \implies l^2 = 2y^2 \).

\[
V = \int_{-2}^2 A(x) \, dx = 2 \int_0^2 A(x) \, dx = 2 \int_0^2 \frac{1}{2} l(t)l(t) \, dx = 2 \int_0^2 y^2 \, dx
\]

\[
= 2 \int_0^2 \frac{1}{4} (36 - 9x^2) \, dx = \frac{9}{2} \int_0^2 (4 - x^2) \, dx
\]

\[
= \frac{9}{2} \left[4x - \frac{1}{3} x^3\right]_0^2 = \frac{9}{2} \left(8 - \frac{8}{3}\right) = 24
\]
40. The cross-section of the base corresponding to the coordinate $y$ has length $x = 1 - y$. The corresponding equilateral triangle with side $s$ has area $A(y) = s^2 \left( \frac{\sqrt{3}}{4} \right) = (1 - y)^2 \left( \frac{\sqrt{3}}{4} \right)$. Therefore,

$$
V = \int_0^1 A(y) \, dy = \int_0^1 (1 - y)^2 \left( \frac{\sqrt{3}}{4} \right) \, dy
= \frac{\sqrt{3}}{4} \int_0^1 (1 - 2y + y^2) \, dy = \frac{\sqrt{3}}{4} \left[ y - y^2 + \frac{1}{3} y^3 \right]_0^1
= \frac{\sqrt{3}}{4} \left( 1 - \frac{1}{3} \right) = \frac{\sqrt{3}}{12}
$$

Or:

$$
\int_0^1 (1 - y)^2 \left( \frac{\sqrt{3}}{4} \right) \, dy = \frac{\sqrt{3}}{4} \int_1^0 u^2 (-du) \quad [u = 1 - y] = \frac{\sqrt{3}}{4} \left[ \frac{1}{3} u^3 \right]_1^0 = \frac{\sqrt{3}}{12}
$$

41. The cross-section of the base corresponding to the coordinate $x$ has length $y = 1 - x$. The corresponding square with side $s$ has area

$$
A(x) = s^2 = (1 - x)^2 = 1 - 2x + x^2.
$$

Therefore,

$$
V = \int_0^1 A(x) \, dx = \int_0^1 (1 - 2x + x^2) \, dx
= \left[ x - x^2 + \frac{1}{3} x^3 \right]_0^1 = (1 - 1 + \frac{1}{3}) - 0 = \frac{1}{3}
$$

Or:

$$
\int_0^1 (1 - x)^2 \, dx = \int_1^0 u^2 (-du) \quad [u = 1 - x] = \left[ \frac{1}{3} u^3 \right]_1^0 = \frac{1}{3}
$$

42. The cross-section of the base corresponding to the coordinate $y$ has length $2x = 2\sqrt{1 - y}$. \(y = 1 - x^2 \iff x = \pm \sqrt{1 - y} \) The corresponding square with side $s$ has area $A(x) = s^2 = (2\sqrt{1 - y})^2 = 4(1 - y)$. Therefore,

$$
V = \int_0^1 A(y) \, dy = \int_0^1 4(1 - y) \, dy = 4 \left[ y - \frac{1}{2} y^2 \right]_0^1 = 4 \left[ (1 - \frac{1}{2}) - 0 \right] = 2.
$$

43. The cross-section of the base $b$ corresponding to the coordinate $x$ has length $1 - x^2$. The height $h$ also has length $1 - x^2$, so the corresponding isosceles triangle has area $A(x) = \frac{1}{2} bh = \frac{1}{2} (1 - x^2)^2$. Therefore,

$$
V = \int_{-1}^1 \frac{1}{2} (1 - x^2)^2 \, dx
= 2 \cdot \frac{1}{2} \int_0^1 (1 - 2x^2 + x^4) \, dx \quad \text{[by symmetry]}
= \left[ x - \frac{2}{3} x^3 + \frac{1}{5} x^5 \right]_0^1 = (1 - \frac{2}{3} + \frac{1}{5}) - 0 = \frac{8}{15}
$$
44. (a) \( V = \int_{-r}^{r} A(x) \, dx = 2 \int_{0}^{r} A(x) \, dx = 2 \int_{0}^{r} \frac{1}{2} h \left( 2 \sqrt{r^2 - x^2} \right) \, dx = 2 h \int_{0}^{r} \sqrt{r^2 - x^2} \, dx \)

(b) Observe that the integral represents one quarter of the area of a circle of radius \( r \), so \( V = 2 h \cdot \frac{1}{4} \pi r^2 = \frac{1}{2} \pi hr^2 \).

45. (a) The radius of the barrel is the same at each end by symmetry, since the function \( y = R - cx^2 \) is even. Since the barrel is obtained by rotating the graph of the function \( y \) about the \( x \)-axis, this radius is equal to the value of \( y \) at \( x = \frac{1}{2} h \), which is \( R - c \left( \frac{1}{2} h \right)^2 = R - d = r \).

(b) The barrel is symmetric about the \( y \)-axis, so its volume is twice the volume of that part of the barrel for \( x > 0 \). Also, the barrel is a volume of rotation, so
\[
V = 2 \int_{0}^{h/2} \pi y^2 \, dx = 2 \pi \int_{0}^{h/2} \left( R - cx^2 \right)^2 \, dx = 2 \pi \left[ R^2 x - \frac{2}{3} Rcx^2 + \frac{1}{6} c^2 x^3 \right]_{0}^{h/2}
\]
\[
= 2 \pi \left( \frac{1}{2} R^2 h - \frac{1}{12} Rch^2 + \frac{1}{180} c^2 h^3 \right)
\]

Trying to make this look more like the expression we want, we rewrite it as \( V = \frac{1}{2} \pi h \left[ 2R^2 + \left( R^2 - \frac{1}{2} Rch^2 + \frac{2}{90} c^2 h^4 \right) \right] \).

But \( R^2 - \frac{1}{2} Rch^2 + \frac{2}{90} c^2 h^4 = (R - \frac{1}{2} c h)^2 - \frac{1}{45} c^2 h^4 = (R - d)^2 - \frac{3}{5} \left( \frac{1}{2} c h \right)^2 = r^2 - \frac{3}{5} d^2 \).

Substituting this back into \( V \), we see that \( V = \frac{1}{2} \pi h \left( 2R^2 + r^2 - \frac{3}{5} d^2 \right) \), as required.

46. (a) \( V = \int_{-1}^{1} \pi \left[ (ax^3 + bx^2 + cx + d) \sqrt{1 - x^2} \right]^2 \, dx \approx 4 \left\{ 5a^2 + 18ac + 3 \left[ 3b^2 + 14bd + 7 \left( c^2 + 5d^2 \right) \right] \right\} \pi \)

(b) \( y = (-0.06x^2 + 0.04x^2 + 0.1x + 0.54)\sqrt{1 - x^2} \) is graphed in the figure. Substitute \( a = -0.06, b = 0.04, c = 0.1, \) and \( d = 0.54 \) in the answer for part (a) to get \( V \approx \frac{3769 \pi}{9375} \approx 1.263 \).
47. (a) The torus is obtained by rotating the circle \((x - R)^2 + y^2 = r^2\) about the \(y\)-axis. Solving for \(x\), we see that the right half of the circle is given by 
\[x = R + \sqrt{r^2 - y^2} = f(y)\]
and the left half by 
\[x = R - \sqrt{r^2 - y^2} = g(y)\].
So
\[V = \pi \int_{-r}^{r} \left\{(f(y))^2 - (g(y))^2\right\} dy\]
\[= 2\pi \int_{0}^{r} \left[\left(R^2 + 2R \sqrt{r^2 - y^2} + r^2 - y^2\right) - \left(R^2 - 2R \sqrt{r^2 - y^2} + r^2 - y^2\right)\right] dy\]
\[= 2\pi R \int_{0}^{r} 4R \sqrt{r^2 - y^2} dy = 8\pi R \int_{0}^{r} \sqrt{r^2 - y^2} dy\]
(b) Observe that the integral represents a quarter of the area of a circle with radius \(r\), so
\[8\pi R \int_{0}^{r} \sqrt{r^2 - y^2} dy = 8\pi R \cdot \frac{1}{4} \pi r^2 = 2\pi^2 r^2 R.\]

48. If we place the \(x\)-axis along the diameter where the planes meet, then the base of the solid is a semicircle with equation \(y = \sqrt{16 - x^2}\), \(-4 \leq x \leq 4\). A cross-section perpendicular to the \(x\)-axis at a distance \(x\) from the origin is a triangle \(ABC\), as shown in the figure, whose base is \(y = \sqrt{16 - x^2}\) and whose height is \(|BC| = y \tan 30^\circ = \sqrt{16 - x^2}/\sqrt{3}\). Thus, the cross-sectional area is 
\[A(x) = \frac{1}{2} \sqrt{16 - x^2} \cdot \frac{1}{\sqrt{3}} \sqrt{16 - x^2} = \frac{16 - x^2}{2\sqrt{3}}\]
and the volume is
\[V = \int_{-4}^{4} A(x) \, dx = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}} \, dx = \frac{1}{\sqrt{3}} \int_{0}^{4} (16 - x^2) \, dx\]
\[= \frac{1}{\sqrt{3}} \left[16x - \frac{1}{3} x^3\right]_{0}^{4} = \frac{128}{3\sqrt{3}}\]

Another method: The cross-sections perpendicular to the \(y\)-axis in the figure are rectangles. The rectangle corresponding to the coordinate \(y\) has a base of length \(2\sqrt{16 - y^2}\) in the \(xy\)-plane and a height of \(\frac{1}{\sqrt{2}}y\), since \(\angle BAC = 30^\circ\) and \(|BC| = \frac{1}{\sqrt{2}}|AB|\). Thus, 
\[A(y) = \frac{2}{\sqrt{2}} y \sqrt{16 - y^2}\]
and
\[V = \int_{0}^{4} A(y) \, dy = \frac{2}{\sqrt{3}} \int_{0}^{4} \sqrt{16 - y^2} \, dy = \frac{2}{\sqrt{3}} \int_{16}^{0} u^{1/2} \left(-\frac{1}{2} \, du\right) \left[u = 16 - y^2\right]
\[= \frac{1}{\sqrt{3}} \int_{0}^{16} u^{1/2} \, du = \frac{1}{\sqrt{3}} \left[\frac{2}{3} u^{3/2}\right]_{0}^{16} = \frac{2}{3\sqrt{3}} (64) = \frac{128}{3\sqrt{3}}\]
49. (a) \( \text{Volume}(S_1) = \int_0^h A(z) \, dz = \text{Volume}(S_2) \) since the cross-sectional area \( A(z) \) at height \( z \) is the same for both solids.

(b) By Cavalieri’s Principle, the volume of the cylinder in the figure is the same as that of a right circular cylinder with radius \( r \) and height \( h \), that is, \( \pi r^2 h \).

50. Each cross-section of the solid \( S \) in a plane perpendicular to the \( x \)-axis is a square (since the edges of the cut lie on the cylinders, which are perpendicular). One-quarter of this square and one-eighth of \( S \) are shown. The area of this quarter-square is \( |PQ|^2 = r^2 - x^2 \). Therefore, \( A(x) = 4(r^2 - x^2) \) and the volume of \( S \) is
\[
V = \int_{-r}^{r} A(x) \, dx = 4 \int_{-r}^{r} (r^2 - x^2) \, dx
\]
\[
= 8(r^2 - x^2) \, dx = 8 \left[ r^2 x - \frac{1}{3} x^3 \right]_0^r = \frac{16}{3} r^3
\]
51. The volume is obtained by rotating the area common to two circles of radius $r$, as shown. The volume of the right half is

$$V_{\text{right}} = \pi \int_0^{r/2} y^2 \, dx = \pi \int_0^{r/2} \left[ r^2 - \left( \frac{1}{2}r + x \right)^2 \right] \, dx$$

$$= \pi \left[ r^2x - \frac{1}{3} \left( \frac{1}{2}r + x \right)^3 \right]_0^{r/2} = \pi \left[ \left( \frac{1}{2}r^2 - \frac{1}{3}r^2 \right) - \left( 0 - \frac{1}{3}r^3 \right) \right] = \frac{5}{24} \pi r^3$$

So by symmetry, the total volume is twice this, or $\frac{5}{12} \pi r^3$.

*Another solution:* We observe that the volume is the twice the volume of a cap of a sphere, so we can use the formula from Exercise 33 with $h = \frac{1}{2}r$: $V = 2 \cdot \frac{1}{2} \pi h^2 (3r - h) = \frac{3}{2} \pi \left( \frac{1}{2}r \right)^2 (3r - \frac{1}{2}r) = \frac{5}{12} \pi r^3$.

52. We consider two cases: one in which the ball is not completely submerged and the other in which it is.

*Case 1: $0 \leq h \leq 10$*  The ball will not be completely submerged, and so a cross-section of the water parallel to the surface will be the shaded area shown in the first diagram. We can find the area of the cross-section at height $x$ above the bottom of the bowl by using the Pythagorean Theorem: $R^2 = 15^2 - (15 - x)^2$ and $r^2 = 5^2 - (x - 5)^2$, so $A(x) = \pi (R^2 - r^2) = 20\pi x$.

The volume of water when it has depth $h$ is then $V(h) = \int_0^h A(x) \, dx = \int_0^h 20\pi x \, dx = [10\pi x^2]_0^h = 10\pi h^2$ cm$^3$, $0 \leq h \leq 10$.

*Case 2: $10 < h \leq 15$*  In this case we can find the volume by simply subtracting the volume displaced by the ball from the total volume inside the bowl underneath the surface of the water. The total volume underneath the surface is just the volume of a cap of the bowl, so we use the formula from Exercise 49: $V_{\text{cap}}(h) = \frac{1}{3} \pi h^2 (45 - h)$. The volume of the small sphere is $V_{\text{ball}} = \frac{4}{3} \pi (5)^3 = \frac{500}{3} \pi$, so the total volume is $V_{\text{cap}} - V_{\text{ball}} = \frac{1}{3} \pi (45h^2 - h^2 - 500)$ cm$^3$.

53. Take the $x$-axis to be the axis of the cylindrical hole of radius $r$.

A quarter of the cross-section through $y$, perpendicular to the $y$-axis, is the rectangle shown. Using the Pythagorean Theorem twice, we see that the dimensions of this rectangle are

$$x = \sqrt{R^2 - y^2} \text{ and } z = \sqrt{r^2 - y^2},$$

so

$$\frac{1}{4} A(y) = xz = \sqrt{r^2 - y^2} \sqrt{R^2 - y^2},$$

and

$$V = \int_{-r}^r A(y) \, dy = \int_{-r}^r 4 \sqrt{r^2 - y^2} \sqrt{R^2 - y^2} \, dy = 8 \int_0^r \sqrt{r^2 - y^2} \sqrt{R^2 - y^2} \, dy$$
54. The line $y = r$ intersects the semicircle $y = \sqrt{R^2 - x^2}$ when $r = \sqrt{R^2 - x^2} \Rightarrow r^2 = R^2 - x^2 \Rightarrow x^2 = R^2 - r^2 \Rightarrow x = \pm \sqrt{R^2 - r^2}$. Rotating the shaded region about the $x$-axis gives us

$$V = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left[ \left( \sqrt{R^2 - x^2} \right)^2 - r^2 \right] dx = 2\pi \int_{0}^{\sqrt{R^2-r^2}} (R^2 - x^2 - r^2) dx \quad \text{[by symmetry]}$$

$$= 2\pi \int_{0}^{\sqrt{R^2-r^2}} \left[ (R^2 - r^2) - x^2 \right] dx = 2\pi \left[ (R^2 - r^2)x - \frac{1}{3}x^3 \right]_{0}^{\sqrt{R^2-r^2}}$$

$$= 2\pi \left[ (R^2 - r^2)^{3/2} - \frac{1}{3} (R^2 - r^2)^{3/2} \right] = 2\pi \cdot \frac{2}{3} (R^2 - r^2)^{3/2} = \frac{4\pi}{3} (R^2 - r^2)^{3/2}$$

Our answer makes sense in limiting cases. As $r \to 0$, $V \to \frac{2}{3}\pi R^3$, which is the volume of the full sphere. As $r \to R$, $V \to 0$, which makes sense because the hole's radius is approaching that of the sphere.
If we were to use the “washed” method, we would first have to locate the local maximum point \((a, b)\) of \(y = x(x - 1)^2\) using the methods of Chapter 4. Then we would have to solve the equation \(y = x(x - 1)^2\) for \(x\) in terms of \(y\) to obtain the functions \(x = g_1(y)\) and \(x = g_2(y)\) shown in the first figure. This step would be difficult because it involves the cubic formula. Finally we would find the volume using

\[
V = \pi \int_0^b \left( [g_1(y)]^3 - [g_2(y)]^3 \right) \, dy.
\]

Using shells, we find that a typical approximating shell has radius \(x\), so its circumference is \(2\pi x\). Its height is \(y\), that is, \(x(x - 1)^2\). So the total volume is

\[
V = \int_0^1 2\pi x \left[ x(x - 1)^2 \right] \, dx = 2\pi \int_0^1 \left( x^5 - 2x^3 + x^2 \right) \, dx = 2\pi \left[ \frac{x^6}{6} - \frac{2x^4}{4} + \frac{x^3}{3} \right]_0^1 = \frac{\pi}{18}
\]

A typical cylindrical shell has circumference \(2\pi x\) and height \(\sin(x^2)\).

\[
V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2) \, dx.
\]

Let \(u = x^2\). Then \(du = 2x \, dx\), so

\[
V = \pi \int_0^{\pi} \sin u \, du = \pi \left[ -\cos u \right]_0^{\pi} = \pi [1 - (-1)] = 2\pi.
\]

For slicing, we would first have to locate the local maximum point \((a, b)\) of \(y = \sin(x^2)\) using the methods of Chapter 4. Then we would have to solve the equation \(y = \sin(x^2)\) for \(x\) in terms of \(y\) to obtain the functions \(x = g_1(y)\) and \(x = g_2(y)\) shown in the second figure. Finally we would find the volume using

\[
V = \pi \int_0^b \left( [g_1(y)]^3 - [g_2(y)]^3 \right) \, dy.
\]

Using shells is definitely preferable to slicing.

3. \(V = \int_0^1 2\pi x \sqrt[4]{x} \, dx = 2\pi \int_0^1 x^{4/3} \, dx = 2\pi \left[ \frac{3}{7} x^{7/3} \right]_0^1 = 2\pi \left( \frac{3}{7} \right) = \frac{6\pi}{7}\)

4. \(V = \int_0^1 2\pi x \cdot x^2 \, dx = 2\pi \int_0^1 x^3 \, dx = 2\pi \left[ \frac{1}{4} x^4 \right]_0^1 = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}\)
5. \( V = \int_0^1 2\pi xe^{-x^2} \, dx \). Let \( u = x^2 \).
   
   Thus, \( du = 2x \, dx \), so
   
   \[
   V = \pi \int_0^1 e^{-u} \, du = \pi \left[ -e^{-u} \right]_0^1 = \pi(1 - 1/e).
   \]

6. \( V = 2\pi \int_0^3 \left\{ x[(3 + 2x - x^2) - (3 - x)] \right\} \, dx 
   = 2\pi \int_0^3 \left[ x(3x - x^2) \right] \, dx 
   = 2\pi \int_0^3 (3x^3 - x^2) \, dx 
   = 2\pi \left[ \frac{3}{4}x^4 - \frac{1}{5}x^3 \right]_0^3 
   = 2\pi \left( \frac{27}{4} - \frac{51}{4} \right) 
   = 2\pi \left( \frac{27}{4} - \frac{51}{4} \right) 
   = \frac{27\pi}{2}.

7. \( x^2 = 6x - 2x^2 \iff 3x^2 - 6x = 0 \iff 3x(x - 2) = 0 \iff x = 0 \text{ or } 2. 
   \)
   
   \[
   V = \int_0^2 2\pi x [(6x - 2x^2) - x^2] \, dx 
   = 2\pi \int_0^2 (-3x^3 + 6x^2) \, dx 
   = 2\pi \left[ -\frac{3}{4}x^4 + \frac{6}{2}x^2 \right]_0^2 
   = 2\pi \left( -12 + 16 \right) = 8\pi.
   \]
8. By slicing:

\[ V = \int_{0}^{1} \pi \left[ \left( \sqrt{y} \right)^2 - (y^2)^2 \right] \, dy = \pi \int_{0}^{1} (y - y^4) \, dy \]

\[ = \pi \left[ \frac{y^3}{3} - \frac{y^5}{5} \right]_{0}^{1} = \pi \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{2}{15} \pi \]

By cylindrical shells:

\[ V = \int_{0}^{1} 2\pi x (\sqrt{x} - x^2) \, dx = 2\pi \int_{0}^{1} (x^{3/2} - x^3) \, dx = 2\pi \left[ \frac{2}{5}x^{5/2} - \frac{1}{4}x^4 \right]_{0}^{1} \]

\[ = 2\pi \left( \frac{2}{5} - \frac{1}{4} \right) = 2\pi \left( \frac{3}{20} \right) = \frac{3}{10} \pi \]

9. \( xy = 1 \Rightarrow x = \frac{1}{y} \), so

\[ V = 2\pi \int_{1}^{3} y \left( \frac{1}{y} \right) \, dy \]

\[ = 2\pi \int_{1}^{3} dy = 2\pi [y]_{1}^{3} \]

\[ = 2\pi (3 - 1) = 4\pi \]

10. \( V = \int_{0}^{1} 2\pi y \sqrt{y} \, dy = 2\pi \int_{0}^{1} y^{3/2} \, dy \)

\[ = 2\pi \left[ \frac{2}{5}y^{5/2} \right]_{0}^{1} = \frac{4}{5} \pi \]
11. \[ V = 2\pi \int_0^8 \left[ y \sqrt{y} - 0 \right] \, dy \]
\[ = 2\pi \int_0^8 y^{4/3} \, dy = 2\pi \left[ \frac{3}{7} y^{7/3} \right]_0^8 \]
\[ = \frac{6\pi}{7} (8^{7/3}) = \frac{6\pi}{7} (2^7) = \frac{768}{7} \pi \]

12. \[ V = 2\pi \int_0^4 \left[ y(4y^2 - y^2) \right] \, dy \]
\[ = 2\pi \int_0^4 (4y^3 - y^4) \, dy \]
\[ = 2\pi \left[ y^4 - \frac{1}{5} y^5 \right]_0^4 = 2\pi (256 - \frac{1024}{5}) \]
\[ = 2\pi \left( \frac{1280}{5} \right) = \frac{512}{5} \pi \]

13. The height of the shell is \( 2 - [1 + (y - 2)^2] = 1 - (y - 2)^2 = 1 - (y^2 - 4y + 4) = -y^2 + 4y - 3 \)
\[ V = 2\pi \int_1^3 y(-y^2 + 4y - 3) \, dy \]
\[ = 2\pi \int_1^3 (-y^3 + 4y^2 - 3y) \, dy \]
\[ = 2\pi \left[ -\frac{1}{4} y^4 + \frac{4}{3} y^3 - \frac{3}{2} y^2 \right]_1^3 \]
\[ = 2\pi \left[ (-\frac{81}{4} + 3\epsilon - \frac{27}{2}) - (-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}) \right] \]
\[ = 2\pi \left( \frac{15}{2} \right) = \frac{15}{2} \pi \]
14. \( V = \int_0^1 2\pi y \left[ 4 - (y - 1)^2 - (3 - y) \right] \, dy \)
\[ = 2\pi \int_0^1 y(-y^2 + 3y) \, dy \]
\[ = 2\pi \int_0^1 (-y^3 + 3y^2) \, dy = 2\pi \left[ -\frac{1}{4}y^4 + y^3 \right]_0^1 \]
\[ = 2\pi \left( -\frac{81}{4} + 27 \right) = 2\pi \left( \frac{27}{4} \right) = \frac{27}{2}\pi \]

15. The shell has radius \( 2 - x \), circumference \( 2\pi(2 - x) \), and height \( x^4 \).
\( V = \int_0^1 2\pi (2-x) x^4 \, dx \)
\[ = 2\pi \int_0^1 (2x^4 - x^5) \, dx \]
\[ = 2\pi \left[ \frac{2}{5}x^5 - \frac{1}{6}x^6 \right]_0^1 \]
\[ = 2\pi \left[ (\frac{2}{5} - \frac{1}{6}) - 0 \right] = 2\pi \left( \frac{7}{30} \right) = \frac{7}{15}\pi \]

16. The shell has radius \( x + 1 \), circumference \( 2\pi(x + 1) \), and height \( \sqrt{x} \).
\( V = \int_0^1 2\pi (x + 1) \sqrt{x} \, dx \)
\[ = 2\pi \int_0^1 (x^{3/2} + x^{1/2}) \, dx \]
\[ = 2\pi \left[ \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} \right]_0^1 \]
\[ = 2\pi \left[ (\frac{2}{5} + \frac{2}{3}) - 0 \right] = 2\pi \left( \frac{16}{15} \right) = \frac{32}{15}\pi \]

17. The shell has radius \( x - 1 \), circumference \( 2\pi(x - 1) \), and height \( (4x - x^2) - 3 = -x^2 + 4x - 3 \).
\( V = \int_1^3 2\pi (x - 1)(-x^2 + 4x - 3) \, dx \)
\[ = 2\pi \int_1^3 (-x^3 + 5x^2 - 7x + 3) \, dx \]
\[ = 2\pi \left[ -\frac{1}{4}x^4 + \frac{5}{3}x^3 - \frac{7}{2}x^2 + 3x \right]_1^3 \]
\[ = 2\pi \left[ \left( -\frac{81}{4} + 45 - \frac{63}{2} + 9 \right) - \left( -\frac{1}{4} + \frac{5}{3} - \frac{7}{2} + 3 \right) \right] \]
\[ = 2\pi \left( \frac{3}{2} \right) = \frac{3}{2}\pi \]
18. The shell has radius $1 - x$, circumference $2\pi(1 - x)$, and height $(2 - x^2) - x^2 = 2 - 2x^2$.

\[ V = \int_{-1}^{1} 2\pi(1 - x)(2 - 2x^2) \, dx \]
\[ = 2\pi(2) \int_{-1}^{1} (1 - x)(1 - x^2) \, dx \]
\[ = 4\pi \int_{-1}^{1} (1 - x - x^2 + x^3) \, dx \]
\[ = 4\pi(2) \int_{0}^{1} (1 - x^2) \, dx \quad \text{[by Theorem 4.5.6]} \]
\[ = 8\pi \left[ x - \frac{1}{3}x^3 \right]_{0}^{1} = 8\pi \left[ (1 - \frac{1}{3}) - 0 \right] = 8\pi \left( \frac{2}{3} \right) = \frac{16}{3} \pi \]

19. The shell has radius $1 - y$, circumference $2\pi(1 - y)$, and height $1 - \frac{y}{\sqrt{y}}$ \[\quad \text{[} y = x^2 \implies x = \sqrt[3]{y} \text{]}\]

\[ V = \int_{0}^{1} 2\pi(1 - y)(1 - y^{1/2}) \, dy \]
\[ = 2\pi \int_{0}^{1} (1 - y - y^{1/2} + y^{1/2}) \, dy \]
\[ = 2\pi \left[ y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2} + \frac{2}{9}y^{3/2} \right]_{0}^{1} \]
\[ = 2\pi \left[ (1 - \frac{1}{2} - \frac{2}{9} + \frac{2}{9}) - 0 \right] \]
\[ = 2\pi \left( \frac{7}{9} \right) = \frac{14}{9} \pi \]

20. The shell has radius $y - (-1) = y + 1$, circumference $2\pi(y + 1)$, and height $\sqrt{y} - y^2$.

\[ V = \int_{0}^{1} 2\pi(y + 1)\left(\sqrt{y} - y^2\right) \, dy \]
\[ = 2\pi \int_{0}^{1} (y^{3/2} + y^{1/2} - y^3 - y^2) \, dy \]
\[ = 2\pi \left[ \frac{2}{9}y^{9/2} + \frac{2}{3}y^{3/2} - \frac{1}{4}y - \frac{1}{3}y^3 \right]_{0}^{1} \]
\[ = 2\pi \left( \frac{2}{9} + \frac{2}{3} - \frac{1}{4} - \frac{1}{3} \right) = 2\pi \left( \frac{28}{54} \right) = \frac{28}{27} \pi \]

21. (a) \[ V = 2\pi \int_{0}^{2} x(xe^{-x}) \, dx = 2\pi \int_{0}^{2} x^2 e^{-x} \, dx \]

(b) \[ V \approx 4.06300 \]
22. (a) \( V = 2\pi \int_0^{\pi/4} \left( \frac{\pi}{2} - x \right) \tan x \, dx \)

(b) \( V \approx 2.25323 \)

23. (a) \( V = 2\pi \int_{-\pi/2}^{\pi/2} (\pi - x)[\cos^4 x - (-\cos^4 x)] \, dx \)

\[ = 4\pi \int_{-\pi/2}^{\pi/2} (\pi - x) \cos^4 x \, dx \]

(b) \( V \approx 46.50942 \)

24. (a) \( x = \frac{2x}{1 + x^2} \Rightarrow x + x^4 = 2x \Rightarrow x^4 - x = 0 \Rightarrow \)

\[ x(x^2 - 1) = 0 \Rightarrow x(x - 1)(x^2 + x + 1) = 0 \Rightarrow x = 0 \text{ or } 1 \]

\( V = 2\pi \int_0^1 [x - (-1)] \left( \frac{2x}{1 + x^2} - x \right) \, dx \)

(b) \( V \approx 2.36164 \)

25. (a) \( V = \int_0^\pi 2\pi (4 - y) \sqrt{\sin y} \, dy \)

(b) \( V \approx 36.57476 \)
26. (a) \[ V = \int_{-3}^{3} 2\pi (5 - y) \left(4 - \sqrt{y^2 + 7}\right) \, dy \]  
   (b) \[ V \approx 163.02712 \]

27. \[ \Delta x = \frac{\pi/4 - 0}{4} = \frac{\pi}{16} \]  
   \[ V = \int_{0}^{\pi/4} 2\pi x \tan x \, dx \approx 2\pi \cdot \frac{\pi}{16} \left(\frac{\pi}{22} + \frac{2\pi}{22} + \frac{5\pi}{22} + \frac{7\pi}{22} \tan \frac{\pi}{22}\right) \approx 1.142 \]

28. (a) Using cylindrical shells, \[ V = \int_{2}^{10} 2\pi x f(x) \, dx = 2\pi \int_{2}^{10} x f(x) \, dx = 2\pi I_1. \]  
   Now use Simpson’s Rule to approximate \(I_1\):  
   \[ I_1 \approx S_8 = \frac{10 - 2}{3}(2f(2) + 4 \cdot 3f(3) + 2 \cdot 4f(4) + 4 \cdot 5f(5) + 2 \cdot 6f(6)) \]  
   \[ + 4 \cdot 7f(7) + 2 \cdot 8f(8) + 4 \cdot 9f(9) + 10f(10)) \]  
   \[ \approx \frac{1}{8} \left[ 2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0) \right] \]  
   \[ = \frac{1}{8}(395.2) \]  
   Thus, \( V \approx 2\pi \cdot \frac{1}{8}(395.2) \approx 827.7 \) or 828 cubic units.

(b) Using disks, \[ V = \int_{2}^{10} \pi [f(x)]^2 \, dx = \pi \int_{2}^{10} [f(x)]^2 \, dx = \pi I_2. \] Now use Simpson’s Rule to approximate \(I_2\):  
   \[ I_2 \approx S_8 = \frac{10 - 2}{3} \left\{ [f(2)]^2 + 4[f(3)]^2 + 2[f(4)]^2 + 4[f(5)]^2 + 2[f(6)]^2 \right. \]  
   \[ + 4[f(7)]^2 + 2[f(8)]^2 + 4[f(9)]^2 + [f(10)]^2 \} \]  
   \[ \approx \frac{1}{8} \left[ (0)^2 + 4(1.5)^2 + 2(1.9)^2 + 4(2.2)^2 + 2(3.0)^2 + 4(3.8)^2 + 2(4.0)^2 + 4(3.1)^2 + (0)^2 \right] \]  
   \[ = \frac{1}{8}(181.78) \]  
   Thus, \( V \approx \pi \cdot \frac{1}{8}(181.78) \approx 190.4 \) or 190 cubic units.

29. \[ \int_{0}^{3} 2\pi x^5 \, dx = 2\pi \int_{0}^{3} x(x^4) \, dx. \] The solid is obtained by rotating the region \( 0 \leq y \leq x^4, 0 \leq x \leq 3 \) about the y-axis using cylindrical shells.

30. \[ 2\pi \int_{0}^{2} \frac{y}{1 + y^2} \, dy = 2\pi \int_{0}^{2} y \left(\frac{1}{1 + y^2}\right) \, dy. \] The solid is obtained by rotating the region \( 0 \leq x \leq \frac{1}{1 + y^2}, 0 \leq y \leq 2 \) about the x-axis using cylindrical shells.
31. \( \int_0^1 2\pi(3 - y)(1 - y^2) \, dy \). The solid is obtained by rotating the region bounded by (i) \( x = 1 - y^2 \), \( x = 0 \), and \( y = 0 \) or (ii) \( x = y^2 \), \( x = 1 \), and \( y = 0 \) about the line \( y = 3 \) using cylindrical shells.

32. \( \int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x) \, dx \). The solid is obtained by rotating the region bounded by (i) \( 0 \leq y \leq \cos x - \sin x \), \( 0 \leq x \leq \frac{\pi}{4} \) or (ii) \( \sin x \leq y \leq \cos x \), \( 0 \leq x \leq \frac{\pi}{4} \) about the line \( x = \pi \) using cylindrical shells.

33. Use shells:
\[
V = \int_2^4 2\pi x (-x^2 + 6x - 8) \, dx = 2\pi \int_2^4 (-x^2 + 6x - 8) \, dx
\]
\[
= 2\pi \left[ -\frac{1}{3} x^4 + 2x^3 - 4x^2 \right]_2^4
\]
\[
= 2\pi \left[ (-64 + 128 - 64) - (-4 + 16 - 16) \right]
\]
\[
= 2\pi(4) = 8\pi
\]

34. Use shells:
\[
V = \int_1^2 2\pi x (-x^2 + 3x - 2) \, dx = 2\pi \int_1^2 (-x^2 + 3x^2 - 2x) \, dx = 2\pi \left[ -\frac{1}{3} x^4 + x^3 - x^2 \right]_1^2
\]
\[
= 2\pi \left[ (-4 + 8 - 4) - \left( -\frac{1}{3} + 1 - 1 \right) \right] = \frac{4\pi}{3}
\]

35. Use washers: \( y^2 - x^2 = 1 \) \( \Rightarrow \) \( y = \pm \sqrt{x^2 + 1} \)
\[
V = \int_{-\sqrt{3}}^{\sqrt{3}} \pi \left[ (2 - 0)^2 - \left( \sqrt{x^2 + 1} \right)^2 \right] \, dx
\]
\[
= 2\pi \int_0^{\sqrt{3}} [4 - (x^2 + 1)] \, dx \quad \text{[by symmetry]}
\]
\[
= 2\pi \int_0^{\sqrt{3}} (3 - x^2) \, dx = 2\pi \left[ 3x - \frac{1}{3} x^3 \right]_0^{\sqrt{3}}
\]
\[
= 2\pi (3\sqrt{3} - \sqrt{3}) = 4\sqrt{3}\pi
\]

36. Use disks: \( y^2 - x^2 = 1 \) \( \Rightarrow \) \( x = \pm \sqrt{y^2 - 1} \)
\[
V = \pi \int_1^2 \left( \sqrt{y^2 - 1} \right)^2 \, dy = \pi \int_1^2 (y^2 - 1) \, dy
\]
\[
= \pi \left[ \frac{1}{3} y^3 - y \right]_1^2 = \pi \left[ \left( \frac{8}{3} - 2 \right) - \left( \frac{1}{3} - 1 \right) \right] = \frac{3}{2}\pi
\]
37. Use disks: 
\[ x^2 + (y - 1)^2 = 1 \quad \Leftrightarrow \quad x = \pm \sqrt{1 - (y - 1)^2} \]

\[ V = \pi \int_0^2 \left[ \sqrt{1 - (y - 1)^2} \right]^2 dy = \pi \int_0^2 (2y - y^2) dy \]

\[ = \pi \left[ y^2 - \frac{1}{3} y^3 \right]_0^2 = \pi \left( 4 - \frac{8}{3} \right) = \frac{4}{3} \pi \]

38. Use shells:

\[ V = \int_1^5 2\pi (y - 1)(1 - (y - 3)^2) dy \]

\[ = 2\pi \int_1^5 (y - 1)(-y^2 + 6y - 5) dy \]

\[ = 2\pi \int_1^5 (-y^3 + 7y^2 - 11y + 5) dy \]

\[ = 2\pi \left[ -\frac{1}{4} y^4 + \frac{7}{3} y^3 - \frac{11}{2} y^2 + 5y \right]_1^5 \]

\[ = 2\pi \left( \frac{279}{12} - \frac{10}{12} \right) = \frac{128}{3} \pi \]

39. Use shells:

\[ V = 2 \int_0^4 2\pi x \sqrt{r^2 - x^2} dx = -2\pi \int_0^r (r^2 - x^2)^{1/2} (-2x) \, dx \]

\[ = \left[ -2\pi \cdot \frac{2}{3} (r^2 - x^2)^{3/2} \right]_0^r = -\frac{4}{3} \pi (0 - r^3) = \frac{4}{3} \pi r^3 \]

40. \( V = \int_{R-r}^{R+r} 2\pi x \cdot 2 \sqrt{r^2 - (x-R)^2} \, dx \)

\[ = \int_{-r}^{r} 4\pi (u + R) \sqrt{r^2 - u^2} \, du \quad \text{[let } u = x - R] \]

\[ = 4\pi R \int_{-r}^{r} \sqrt{r^2 - u^2} \, du + 4\pi \int_{-r}^{r} u \sqrt{r^2 - u^2} \, du \]

The first integral is the area of a semicircle of radius \( r \), that is, \( \frac{1}{2} \pi r^2 \),

and the second is zero since the integrand is an odd function. Thus,

\[ V = 4\pi R \left( \frac{1}{2} \pi r^2 \right) + 4\pi \cdot 0 = 2\pi^2 R r^2. \]

41. \( V = 2\pi \int_0^r x \left( -\frac{h}{r} x + h \right) dx = 2\pi h \int_0^r \left( -\frac{x^2}{r} + x \right) \, dx \)

\[ = 2\pi h \left[ -\frac{x^2}{3r} + \frac{x^2}{2} \right]_0^r = 2\pi h \left( -\frac{r^2}{3r} + \frac{r^2}{2} \right) = \frac{\pi r^2 h}{3} \]
42. By symmetry, the volume of a napkin ring obtained by drilling a hole of radius \( r \) through a sphere with radius \( R \) is twice the volume obtained by rotating the area above the \( x \)-axis and below the curve \( y = \sqrt{R^2 - x^2} \) (the equation of the top half of the cross-section of the sphere), between \( x = r \) and \( x = R \), about the \( y \)-axis. This volume is equal to

\[
2 \int_{\text{inner radius}}^{\text{outer radius}} 2\pi rh \, dx = 2 \cdot 2\pi \int_{r}^{R} x \sqrt{R^2 - x^2} \, dx = 4\pi \left[ -\frac{1}{3} (R^3 - x^3)^{2/3} \right]_{r}^{R} = \frac{4}{3} \pi (R^3 - r^3)^{3/2}
\]

But by the Pythagorean Theorem, \( R^2 - r^2 = \left( \frac{1}{2}h \right)^2 \), so the volume of the napkin ring is \( \frac{4}{3} \pi \left( \frac{1}{2}h \right)^2 = \frac{1}{6} \pi h^3 \), which is independent of both \( R \) and \( r \); that is, the amount of wood in a napkin ring of height \( h \) is the same regardless of the size of the sphere used. Note that most of this calculation has been done already, but with more difficulty, in Exercise 7.2.54.

*Another solution:* The height of the missing cap is the radius of the sphere minus half the height of the cut-out cylinder, that is, \( R - \frac{1}{2}h \). Using Exercise 7.2.33,

\[
V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}} = \frac{4}{3} \pi R^3 - \pi r^2 h - 2 \cdot \frac{1}{6} \pi \left( 3R - (R - \frac{1}{2}h) \right) = \frac{1}{6} \pi h^3
\]

43. Using the formula for volumes of rotation and the figure, we see that

\[
\text{Volume} = \int_{a}^{b} \pi b^2 \, dy - \int_{c}^{d} \pi a^2 \, dy - \int_{c}^{d} \pi \left[ f^{-1}(y) \right]^2 \, dy = \pi b^2 d - \pi a^2 c - \pi \int_{a}^{b} x^2 f'(x) \, dx
\]

Let \( y = f(x) \), which gives \( dy = f'(x) \, dx \) and \( f^{-1}(y) = x \), so that \( V = \pi b^2 d - \pi a^2 c - \pi \int_{a}^{b} x^2 f'(x) \, dx \).

Now integrate by parts with \( u = x^2 \) and \( dv = f'(x) \, dx \Rightarrow du = 2x \, dx \), \( v = f(x) \), and

\[
\int_{a}^{b} x^2 f'(x) \, dx = \left[ x^2 f(x) \right]_{a}^{b} - \int_{a}^{b} 2x f(x) \, dx = b^2 f(b) - a^2 f(a) - \int_{a}^{b} 2x f(x) \, dx
\]

but \( f(a) = c \) and \( f(b) = d \) \Rightarrow

\[
V = \pi b^2 d - \pi a^2 c - \pi \left[ b^2 d - a^2 c - \int_{a}^{b} 2x f(x) \, dx \right] = \int_{a}^{b} 2\pi x f(x) \, dx.
\]
1. \( y = 2 - 3x \implies L = \int_{-2}^{2} \sqrt{1 + (dy/dx)^2} \, dx = \int_{-2}^{2} \sqrt{1 + (-3)^2} \, dx = \sqrt{10} \, [1 - (-2)] = 3\sqrt{10}. \)

The arc length can be calculated using the distance formula, since the curve is a line segment, so

\[ L = \text{distance from } (-2, 8) \text{ to } (1, -1) = \sqrt{[1 - (-2)]^2 + [(1-1)-8]^2} = \sqrt{90} = 3\sqrt{10}. \]

2. Using the arc length formula with \( y = \sqrt{4 - x^2} \implies \frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}}, \) we get

\[ L = \int_{0}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{0}^{2} \sqrt{1 + \frac{x^2}{4 - x^2}} \, dx = \int_{0}^{2} \frac{2 \, dx}{\sqrt{4 - x^2}} = 2 \lim_{t \to 2} \int_{0}^{t} \frac{dx}{\sqrt{t^2 - x^2}} = 2 \lim_{t \to 2} \left[ \sin^{-1}(x/2) \right]_{0}^{t} = 2 \lim_{t \to 2} \left[ \sin^{-1}(t/2) - \sin^{-1}0 \right] = 2\left(\frac{\pi}{2} - 0\right) = \pi. \]

The curve is a quarter of a circle with radius 2, so the length of the arc is \( \frac{\pi}{4}(2 \pi \cdot 2) = \pi, \) as above.

3. \( y = \sin x \implies \frac{dy}{dx} = \cos x \implies 1 + (dy/dx)^2 = 1 + \cos^2 x. \) So \( L = \int_{0}^{\pi} \sqrt{1 + \cos^2 x} \, dx \approx 3.8202. \)

4. \( y = xe^{-x} \implies \frac{dy}{dx} = xe^{-x} + e^{-x}(1) = e^{-x}(1 - x) \implies 1 + (dy/dx)^2 = 1 + [e^{-x}(1 - x)]^2. \)

So \( L = \int_{0}^{2} \sqrt{1 + e^{-2x}(1-x)^2} \, dx \approx 2.1024. \)

5. \( x = \sqrt{y} - y \implies dx/dy = 1/(2\sqrt{y}) - 1 \implies 1 + (dx/dy)^2 = \left(\frac{1}{2\sqrt{y}} - 1\right)^2. \)

So \( L = \int_{1}^{4} \sqrt{1 + \left(\frac{1}{2\sqrt{y}} - 1\right)^2} \, dy \approx 3.6095. \)

6. \( x = y^2 - 2y \implies dx/dy = 2y - 2 \implies 1 + (dx/dy)^2 = 1 + (2y - 2)^2. \) So \( L = \int_{0}^{2} \sqrt{1 + (2y - 2)^2} \, dy \approx 2.9579. \)

7. \( y = 1 + 6x^{2/3} \implies \frac{dy}{dx} = 9x^{1/3} \implies 1 + (dy/dx)^2 = 1 + 18x. \) So

\[ L = \int_{0}^{1} \sqrt{1 + 81x} \, dx = \int_{1}^{82} u^{1/2} \left( \frac{1}{81} \, du \right) = \left[ \frac{1}{81} \cdot \frac{2}{3} u^{3/2} \right]_{1}^{82} = \frac{2}{243} (82\sqrt{82} - 1) \]

8. \( y^2 = 4(x + 4)^2, \ y > 0 \implies y = 2(x + 4)^{1/2} \implies \frac{dy}{dx} = 3(x + 4)^{1/2} \implies 1 + (dy/dx)^2 = 1 + 9(x + 4) = 9x + 37. \) So

\[ L = \int_{0}^{5} \sqrt{9x + 37} \, dx = \int_{37}^{55} u^{1/2} \left( \frac{1}{9} \, du \right) = \frac{1}{9} \cdot \frac{2}{3} \left[ u^{3/2} \right]_{37}^{55} = \frac{2}{27} \left( 55\sqrt{55} - 37\sqrt{37} \right). \]
9. \[ y = \frac{x^3}{3} + \frac{1}{4x} \Rightarrow y' = x^2 - \frac{1}{4x^2} \Rightarrow \]

\[ 1 + (y')^2 = 1 + \left( x^2 - \frac{1}{2} + \frac{1}{16x^4} \right) = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left( x^2 + \frac{1}{4x^2} \right)^2. \]

So

\[ L = \int_1^2 \sqrt{1 + (y')^2} \, dx = \int_1^2 \left | x^2 + \frac{1}{4x^2} \right | \, dx = \int_1^2 \left ( x^2 + \frac{1}{4x^2} \right ) \, dx \]

\[ = \left [ \frac{1}{3}x^3 - \frac{1}{4x} \right ]_1^2 = \left ( \frac{8}{3} - \frac{1}{8} \right ) - \left ( \frac{1}{3} - \frac{1}{4} \right ) = \frac{7}{3} + \frac{1}{8} = \frac{59}{24} \]

10. \[ x = \frac{y^4}{8} + \frac{1}{4y^2} \Rightarrow \frac{dx}{dy} = \frac{1}{2}y^3 - \frac{1}{2}y^{-3} \Rightarrow \]

\[ 1 + \left ( \frac{dx}{dy} \right )^2 = 1 + \frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4}y^{-6} = \frac{1}{4}y^6 + \frac{1}{2} + \frac{1}{4}y^{-6} = \left ( \frac{1}{2}y^3 + \frac{1}{2}y^{-3} \right )^2. \]

So

\[ L = \int_1^2 \sqrt{\left ( \frac{1}{2}y^3 + \frac{1}{2}y^{-3} \right )^2} \, dy = \int_1^2 \left ( \frac{1}{2}y^3 + \frac{1}{2}y^{-3} \right ) \, dy = \left [ \frac{1}{8}y^4 - \frac{1}{4}y^{-2} \right ]_1^2 = (2 - \frac{1}{16}) - (\frac{1}{8} - \frac{1}{4}) \]

\[ = 2 + \frac{1}{16} = \frac{33}{16}. \]

11. \[ x = \frac{1}{2} \sqrt{y} (y - 3) = \frac{1}{2}y^{1/2} - y^{1/2} \Rightarrow \frac{dx}{dy} = \frac{1}{2}y^{-1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \]

\[ 1 + \left ( \frac{dx}{dy} \right )^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left ( \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \right )^2. \]

So

\[ L = \int_1^2 \left ( \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \right ) \, dy = \frac{1}{2} \left [ \frac{2}{3}y^{3/2} + 2y^{1/2} \right ]_1^2 = \frac{1}{2} \left [ (\frac{2}{3} \cdot 27 + 2 \cdot 3) - (\frac{2}{3} \cdot 1 + 2 \cdot 1) \right ] \]

\[ = \frac{1}{2} \left ( 24 - \frac{8}{3} \right ) = \frac{1}{2} \left ( \frac{64}{3} \right ) = \frac{32}{3}. \]

12. \[ y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = -\tan x \Rightarrow 1 + \left ( \frac{dy}{dx} \right )^2 = 1 + \tan^2 x = \sec^2 x. \]

So

\[ L = \int_0^\pi \sqrt{\sec^2 x} \, dx = \int_0^\pi \sec x \, dx = [\ln |\sec x + \tan x|]_0^\pi = \ln(2 + \sqrt{3}) - \ln(1 + 0) = \ln(2 + \sqrt{3}). \]

13. \[ y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \Rightarrow 1 + \left ( \frac{dy}{dx} \right )^2 = 1 + \tan^2 x = \sec^2 x, \]

so

\[ L = \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx = \int_0^{\pi/4} |\sec x| \, dx = \int_0^{\pi/4} \sec x \, dx = \left [ \ln(\sec x + \tan x) \right ]_0^{\pi/4} \]

\[ = \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1) \]

14. \[ y = 3 + \frac{1}{2} \cosh 2x \Rightarrow y' = \sinh 2x \Rightarrow 1 + \left ( \frac{dy}{dx} \right )^2 = 1 + \sinh^2(2x) = \cosh^2(2x). \]

So

\[ L = \int_0^1 \sqrt{\cosh^2(2x)} \, dx = \int_0^1 \cosh 2x \, dx = \left [ \frac{1}{2} \sinh 2x \right ]_0^1 = \frac{1}{2} \sinh 2 - 0 = \frac{1}{2} \sinh 2. \]
15. \( y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \Rightarrow y' = \frac{1}{2}x - \frac{1}{2x} \Rightarrow 1 + (y')^2 = 1 + \left(\frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}\right) = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2 \).

So

\[
L = \int_1^2 \sqrt{1 + (y')^2} \, dx = \int_1^2 \left| \frac{1}{2}x + \frac{1}{2x} \right| \, dx = \int_1^2 \left(\frac{1}{2}x + \frac{1}{2x}\right) \, dx
\]

\[= \left[ \frac{1}{4}x^2 + \frac{1}{2}\ln |x| \right]_1^2 = \left( 1 + \frac{1}{2}\ln 2 \right) - \left( \frac{1}{4} + 0 \right) = \frac{3}{4} + \frac{1}{2}\ln 2 \]

16. \( y = \sqrt{x - x^2} + \sin^{-1}\left(\sqrt{x}\right) \Rightarrow \frac{dy}{dx} = \frac{1 - 2x}{2\sqrt{x - x^2}} + \frac{1}{2\sqrt{x}} \sqrt{1 - x} = \frac{2 - 2x}{2\sqrt{x - x^2}} = \sqrt{\frac{1 - x}{x}} \Rightarrow \)

\[1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{1 - x}{x} = 1 \text{. The curve has endpoints } (0, 0) \text{ and } (1, \frac{\pi}{2}) \text{, so } L = \int_0^1 \sqrt{\frac{1}{x}} \, dx = \left[ 2\sqrt{x} \right]_0^1 = 2. \]

17. \( y = \ln(1 - x^2) \Rightarrow y' = \frac{1}{1 - x^2} \cdot (-2x) \Rightarrow \)

\[1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \frac{1 + 2x^2 + x^4}{(1 - x^2)^2} = \frac{(1 + x^2)^2}{(1 - x^2)^2} \Rightarrow \]

\[\sqrt{1 + \left( \frac{dy}{dx} \right)^2} = \sqrt{\frac{1 + x^2}{1 - x^2}} = \frac{1 + x^2}{1 - x^2} = -1 + \frac{2}{1 - x^2} \quad \text{[by division]} = -1 + \frac{1}{1 + x} + \frac{1}{1 - x} \quad \text{[partial fractions].} \]

So \( L = \int_0^{1/2} \left( -1 + \frac{1}{1 + x} + \frac{1}{1 - x} \right) \, dx = \left[ -x + \ln |1 + x| - \ln |1 - x| \right]_0^{1/2} = \left( -\frac{1}{2} + \ln \frac{3}{2} - \ln \frac{1}{2} \right) - 0 = \ln 3 - \frac{1}{2}. \)

18. \( y = 1 - e^{-x} \Rightarrow y' = -(e^{-x}) = e^{-x} \Rightarrow 1 + (dy/dx)^2 = 1 + e^{-2x}. \)

So

\[L = \int_0^2 \sqrt{1 + e^{-2x}} \, dx = \int_1^{e^{-2}} \sqrt{1 + u^2} \left(-\frac{1}{u} \, du \right) \quad [u = e^{-x}]\]

\[= \ln \left[ \frac{1 + \sqrt{1 + u^2}}{u} \right]_1^{e^{-2}} = \ln \left[ \frac{1 + \sqrt{1 + e^{-4}}}{e^{-2}} \right] - \ln \left[ \frac{1 + \sqrt{2}}{1} \right] + \sqrt{2} \]

\[= \ln \left( 1 + \sqrt{1 + e^{-4}} \right) - \ln e^{-2} - \sqrt{1 + e^{-4}} - \ln \left( 1 + \sqrt{2} \right) + \sqrt{2} \]

\[= \ln \left( 1 + \sqrt{1 + e^{-4}} \right) + 2 - \sqrt{1 + e^{-4}} - \ln \left( 1 + \sqrt{2} \right) + \sqrt{2} \]

19. From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points \((1, 2), (1, 12), \) and \((2, 12)\). This length is about \(\sqrt{10^2 + 1^2} \approx 10\), so we might estimate the length to be 10.

\( y = x^2 + x^2 \Rightarrow y' = 2x + 3x^2 \Rightarrow 1 + (y')^2 = 1 + (2x + 3x^2)^2. \)

So \( L = \int_1^2 \sqrt{1 + (2x + 3x^2)^2} \, dx \approx 10.0556. \)
20. From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points (1, 1), \((\frac{\pi}{2}, 1)\), and \((\frac{\pi}{2}, \frac{\pi}{2})\). This length is about \(\sqrt{\left(\frac{\pi}{2}\right)^2 + (\frac{\pi}{2} - 1)^2} \approx 1.7\), so we might estimate the length to be 1.7. 

\[ y = x + \cos x \quad \Rightarrow \quad y' = 1 - \sin x \quad \Rightarrow \]

\[ 1 + (y')^2 = 1 + (1 - \sin x)^2. \] So

\[ L = \int_0^{\pi/2} \sqrt{1 + (1 - \sin x)^2} \, dx \approx 1.7294. \]

21. \[ y = x \sin x \quad \Rightarrow \quad \frac{dy}{dx} = x \cos x + (\sin x)(1) \quad \Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + (x \cos x + \sin x)^2. \]

Let \[ f(x) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (x \cos x + \sin x)^2}. \text{ Then } L = \int_0^{2\pi} f(x) \, dx. \text{ Since } n = 10, \Delta x = \frac{2\pi}{10} = \frac{\pi}{5}. \]

\[ L \approx S_{10} = \frac{\pi}{5} \left[ f(0) + 4f\left(\frac{\pi}{5}\right) + 2f\left(\frac{2\pi}{5}\right) + 4f\left(\frac{3\pi}{5}\right) + 2f\left(\frac{4\pi}{5}\right) + 4f\left(\frac{5\pi}{5}\right) + 2f\left(\frac{6\pi}{5}\right) + 4f\left(\frac{7\pi}{5}\right) + 2f\left(\frac{8\pi}{5}\right) + f(2\pi) \right] \]

\[ \approx 15.498085 \]

The value of the integral produced by a calculator is 15.374568 (to six decimal places).

22. \[ x = y + \sqrt{y} \quad \Rightarrow \quad \frac{dx}{dy} = 1 + \frac{1}{2\sqrt{y}} \quad \Rightarrow \quad 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(1 + \frac{1}{2\sqrt{y}}\right)^2 = 2 + \frac{1}{\sqrt{y}} + \frac{1}{4y}. \]

Let \[ g(y) = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \text{ Then } L = \int_1^2 g(y) \, dy. \text{ Since } n = 10, \Delta y = \frac{2 - 1}{10} = \frac{1}{10}. \]

\[ L \approx S_{10} = \frac{1/10}{2}[g(1) + 4g(1.1) + 2g(1.2) + 4g(1.3) + 2g(1.4) + 4g(1.5) + 2g(1.6) + 4g(1.7) + 2g(1.8) + 4g(1.9) + g(2)] \approx 1.732215, \]

which is the same value of the integral produced by a calculator to six decimal places.

23. \[ y = \ln(1 + x^2) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{1 + x^2} \cdot 2x \quad \Rightarrow \quad L = \int_0^5 f(x) \, dx, \text{ where } f(x) = \sqrt{1 + 9x^4/(1 + x^2)^2}. \text{ Since } n = 10, \]

\[ \Delta x = \frac{5 - 0}{10} = \frac{1}{2}. \text{ Now} \]

\[ L \approx S_{10} = \frac{1/2}{2}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)] \]

\[ \approx 7.094570 \]

The value of the integral produced by a calculator is 7.118819 (to six decimal places).
24. \( y = x \ln x \Rightarrow \frac{dy}{dx} = 1 + \ln x \). Let \( f(x) = \sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + (1 + \ln x)^2} \).

Then \( L = \int_1^2 f(x) \, dx \). Since \( n = 10 \), \( \Delta x = \frac{2 - 1}{10} = \frac{1}{5} \). Now

\[
L \approx S_{10} = \frac{1}{2} \sum_{k=1}^{10} [f(1) + 4f(1.2) + 2f(1.4) + 4f(1.6) + 2f(1.8) + 4f(2) + 2f(2.2) + 4f(2.4) + 2f(2.6) + 4f(2.8) + f(3)] \approx 3.869618
\]

The value of the integral produced by a calculator is 3.869617 (to six decimal places).

25. \( x = \ln (1 - y^2) \Rightarrow \frac{dx}{dy} = \frac{-2y}{1 - y^2} \Rightarrow 1 + \left( \frac{dx}{dy} \right)^2 = 1 + \frac{4y^2}{(1 - y^2)^2} = \frac{(1 + y^2)^2}{(1 - y^2)^2} \). So

\[
L = \int_0^{1/2} \sqrt{\frac{(1 + y^2)^2}{(1 - y^2)^2}} \, dy = \int_0^{1/2} \frac{1 + y^2}{1 - y^2} \, dy = \ln 3 - \frac{1}{2} \quad \text{[from a CAS]}
\]

26. \( y = x^{4/3} \Rightarrow \frac{dy}{dx} = \frac{4}{3} x^{1/3} \Rightarrow 1 + (\frac{dy}{dx})^2 = 1 + \frac{16}{9} x^{2/3} \Rightarrow \)

\[
L = \int_0^1 \sqrt{1 + \frac{16}{9} x^{2/3}} \, dx = \int_0^{4/3} \sqrt{1 + u^2} \frac{81}{64} u^2 \, du \quad \left[ u = \frac{4}{3} x^{1/3}, du = \frac{4}{9} x^{-2/3} \, dx, \quad dx = \frac{9}{4} x^{2/3} \, du = \frac{27}{4} u^2 \, du \right]
\]

\[
= \frac{81}{64} \left[ \frac{1}{2} u(1 + u^2) \sqrt{1 + u^2} - \frac{1}{8} \ln(u + \sqrt{1 + u^2}) \right]_0^{4/3} = \frac{81}{64} \left[ \frac{1}{2} \left(1 + \frac{256}{81}\right) \sqrt{\frac{256}{81}} - \frac{1}{8} \ln \left( \frac{9}{8} + \sqrt{\frac{256}{81}} \right) \right]
\]

\[
= \frac{81}{64} \left[ \frac{1}{2} \cdot \frac{41}{9} - \frac{1}{8} \ln 3 \right] = \frac{295}{128} - \frac{81}{512} \ln 3 \approx 1.4277586
\]

27. \( y^{3/2} = 1 - x^{3/2} \Rightarrow y = (1 - x^{3/2})^{3/2} \Rightarrow \)

\[
\frac{dy}{dx} = \frac{3}{2} \left(1 - x^{3/2}\right)^{1/2} \left( -\frac{3}{2} x^{-1/2} \right) = -x^{-1/2} \left(1 - x^{3/2}\right)^{1/2} \Rightarrow \]

\[
\left( \frac{dy}{dx} \right)^2 = x^{-2/3} (1 - x^{3/2}) = x^{-2/3} - 1. \quad \text{Thus}
\]

\[
L = 4 \int_0^1 \sqrt{1 + (x^{-2/3} - 1)} \, dx = 4 \int_0^1 x^{-1/3} \, dx = 4 \lim_{\epsilon \to 0^+} \left[ \frac{3}{2} x^{2/3} \right]_\epsilon = 6.
\]
28. (a) 
\[ \frac{dy}{dx} \]

(b) \[ y = x^{2/3} \Rightarrow 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \frac{2}{3}x^{-1/3} \right)^2 = 1 + \frac{4}{9}x^{-2/3}. \] So \( L = \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}} \, dx \) [an improper integral].

\[ x = y^{3/2} \Rightarrow 1 + \left( \frac{dx}{dy} \right)^2 = 1 + \left( \frac{3}{2}y^{1/2} \right)^2 = 1 + \frac{9}{4}y. \] So \( L = \int_0^1 \sqrt{1 + \frac{9}{4}y} \, dy. \)

The second integral equals \( \frac{4}{3} \cdot \frac{3}{2} \left[ \left(1 + \frac{9}{4}y \right)^{3/2} \right]_0 = \frac{3}{27} \left[ \frac{3}{4} \right] = \frac{13}{27} - \frac{8}{27}. \)

The first integral can be evaluated as follows:

\[ \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}} \, dx = \lim_{\epsilon \to 0+} \int_{\epsilon}^1 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}} \, dx = \frac{u + 4}{18} \left[ \frac{2}{3} \left( \frac{u + 4}{3} \right)^{3/2} \right]_0^9 = \frac{1}{27} \left( 13^{3/2} - 4^{3/2} \right) = \frac{13\sqrt{13} - 8}{27} \]

(c) \( L = \) length of the arc of this curve from \((-1, 1)\) to \((8, 4)\)

\[ = \int_0^1 \sqrt{1 + \frac{9}{4}y} \, dy + \int_0^4 \sqrt{1 + \frac{9}{4}y} \, dy = \frac{13\sqrt{13} - 8}{27} + \frac{8}{27} \left[ \frac{3}{4} \left( \frac{3}{4} \right)^{3/2} \right]_1^4 = \frac{13\sqrt{13} + 80\sqrt{10} - 16}{27} \]

from part (b)

29. \( y = 2x^{3/2} \Rightarrow y' = 3x^{1/2} \Rightarrow 1 + (y')^2 = 1 + 9x. \) The arc length function with starting point \( P_0(1, 2) \) is

\[ s(x) = \int_1^x \sqrt{1 + 9t} \, dt = \left[ \frac{3}{16} \left( 1 + 9t \right)^{3/2} \right]_1^x = \frac{3}{16} \left[ (1 + 9x)^{3/2} - 10 \sqrt{10} \right]. \]

30. (a) \( y = f(x) = \ln(\sin x) \Rightarrow y' = \frac{1}{\sin x} \cdot \cos x = \cot x \Rightarrow 1 + (y')^2 = 1 + \cot^2 x = \csc^2 x \Rightarrow \)

\( \sqrt{1 + (y')^2} = \sqrt{\csc^2 x} = |\csc x|. \) Therefore,

\[ s(x) = \int_{\pi/2}^x \sqrt{1 + [f'(t)]^2} \, dt = \int_{\pi/2}^x \csc t \, dt = \left[ \ln |\csc t - \cot t| \right]_{\pi/2}^x = \ln |\csc x - \cot x| - \ln |1 - 0| = \ln(\csc x - \cot x) \]

(b) Note that \( s \) is increasing on \((0, \pi)\) and that \( x = 0 \) and \( x = \pi \) are vertical asymptotes for both \( f \) and \( s \).
31. \[ y = \sin^{-1} x + \sqrt{1 - x^2} \implies y' = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \implies \]
\[ 1 + (y')^2 = 1 + \left( \frac{1-x}{\sqrt{1-x^2}} \right)^2 = 1 - \frac{x^2}{1-x^2} = \frac{2}{1-x^2} = \frac{2(1-x)}{(1+x)(1-x)} = \frac{2}{1+x} \implies \]
\[ \sqrt{1 + (y')^2} = \sqrt{1 + \frac{2}{1+x}} = \sqrt{\frac{2}{1+x}}. \text{ Thus, the arc length function with starting point } (0, 1) \text{ is given by} \]
\[ s(x) = \int_0^x \sqrt{1 + (f'(t))^2} \, dt = \int_0^x \frac{\sqrt{2}}{1+t} \, dt = \sqrt{2} \left[ \sqrt{1+t} \right]_0^x = 2 \sqrt{2} (\sqrt{1+x} - 1). \]

32. \[ y = 50 - \frac{1}{10} (x - 15)^2 \implies y' = -\frac{1}{5} (x - 15) \implies 1 + (y')^2 = 1 + \frac{1}{25} (x - 15)^2, \text{ so the distance traveled by the kite is} \]
\[ L = \int_0^{25} \sqrt{1 + \frac{1}{25} (x - 15)^2} \, dx = \int_0^{25} \sqrt{1 + u^2} \, (5 \, du) \quad [u = \frac{1}{5} (x - 15), du = \frac{1}{5} \, dx] \]
\[ \overset{25}{=} 5 \int \left[ \frac{1}{2} u \sqrt{1+u^2} + \frac{1}{2} \ln(u + \sqrt{1+u^2}) \right]_0^{25} = 5 \left[ \sqrt{5} + \frac{1}{2} \ln(2 + \sqrt{5}) + \frac{1}{2} \sqrt{10} - \frac{1}{2} \ln(-3 + \sqrt{10}) \right] \]
\[ = 5\sqrt{5} + \frac{1}{2} \sqrt{10} + \frac{1}{2} \ln \left( \frac{2 + \sqrt{5}}{-3 + \sqrt{10}} \right) \approx 43.1 \text{ m} \]

33. The prey hits the ground when \( y = 0 \) \( \implies 180 - \frac{1}{45} x^2 = 0 \implies x^2 = 45 \cdot 180 \implies x = \sqrt{8100} = 90, \]
\[ \text{since } x \text{ must be positive. } y' = -\frac{2}{45}x \implies 1 + (y')^2 = 1 + \frac{4}{45^2} x^2, \text{ so the distance traveled by the prey is} \]
\[ L = \int_0^{90} \sqrt{1 + \frac{4}{45^2} x^2} \, dx = \int_0^{45} \sqrt{1 + u^2} \left( \frac{45}{u} \, du \right) \quad [u = \frac{x}{45}, du = \frac{1}{45} \, dx] \]
\[ \overset{90}{=} \frac{45}{2} \int \left[ \frac{1}{2} u \sqrt{1+u^2} + \frac{1}{2} \ln(u + \sqrt{1+u^2}) \right]_0^{45} = \frac{45}{2} \left[ 2 \sqrt{17} + \frac{1}{2} \ln(4 + \sqrt{17}) \right] = 45 \sqrt{17} + \frac{45}{4} \ln(4 + \sqrt{17}) \approx 209.1 \text{ m} \]

34. (a) \( y = c + a \cosh \left( \frac{x}{a} \right) \implies \; y' = \sinh \left( \frac{x}{a} \right) \implies \; 1 + (y')^2 = 1 + \sinh^2 \left( \frac{x}{a} \right) = \cosh^2 \left( \frac{x}{a} \right). \text{ So} \]
\[ L = \int_0^b \sqrt{\cosh^2 \left( \frac{x}{a} \right)} \, dx = 2 \int_0^b \cosh \left( \frac{x}{a} \right) \, dx = 2 \left[ a \sinh \left( \frac{x}{a} \right) \right]_0^b = 2a \sinh \left( \frac{b}{a} \right). \]

(b) At \( x = 0, y = c + a, \text{ so } c + a = 9. \text{ The poles are } 20 \text{ m apart, so } b = 10, \)
\[ \text{and } L = 20.4 \implies 20.4 = 2a \sinh \left( \frac{b}{a} \right) \text{ [from part (a)]. From the figure,} \]
\[ \text{we see that } y = 20.4 \text{ intersects } y = 2x \sinh(10/x) \text{ at } x \approx 28.95 \text{ for } x > 0. \]
\[ \text{So } a \approx 28.95 \text{ and the wire should be attached at a distance of} \]
\[ y = c + a \cosh(10/a) = 9 - a + a \cosh(10/a) \approx 10.74 \text{ m above the ground.} \]
35. The sine wave has amplitude 2 and period 30, since it goes through two periods in a distance of 60 cm, so its equation is 
\[ y = 2 \sin\left(\frac{2\pi}{30}x\right) = 2 \sin\left(\frac{\pi}{15}x\right) \]. The width \( w \) of the flat metal sheet needed to make the panel is the arc length of the sine curve from \( x = 0 \) to \( x = 60 \). We set up the integral to evaluate \( w \) using the arc length formula with \( \frac{dy}{dx} = \frac{2\pi}{15} \cos\left(\frac{\pi}{15}x\right) \):
\[ L = \int_0^{60} \sqrt{1 + \left[\frac{2\pi}{15} \cos\left(\frac{\pi}{15}x\right)\right]^2} \, dx \]. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that \( L \approx 62.55 \) cm.

36. By symmetry, the length of the curve in each quadrant is the same, so we’ll find the length in the first quadrant and multiply by 4.
\[ x^{2k} + y^{2k} = 1 \quad \Rightarrow \quad y^{2k} = 1 - x^{2k} \quad \Rightarrow \quad y = (1 - x^{2k})^{1/(2k)} \]
(in the first quadrant), so we use the arc length formula with
\[ \frac{dy}{dx} = \frac{1}{2k} (1 - x^{2k})^{1/(2k) - 1} (-2kx^{2k-1}) = -x^{2k-1}(1 - x^{2k})^{1/(2k) - 1} \]
The total length is therefore
\[ L_{2k} = 4 \int_0^1 \sqrt{1 + [-x^{2k-1}(1 - x^{2k})^{1/(2k) - 1}]^2} \, dx = 4 \int_0^1 \sqrt{1 + x^{2(2k-1)}(1 - x^{2k})^{1/(2k) - 2}} \, dx \]
Now from the graph, we see that as \( k \) increases, the “corners” of these fat circles get closer to the points \((\pm 1, \pm 1)\) and \((\pm 1, \mp 1)\), and the “edges” of the fat circles approach the lines joining these four points. It seems plausible that as \( k \to \infty \), the total length of the fat circle with \( n = 2k \) will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as \( k \to \infty \) of the equation of the fat circle in the first quadrant: \( \lim_{k \to \infty} (1 - x^{2k})^{1/(2k)} = 1 \) for \( 0 \leq x < 1 \). So we guess that \( \lim_{k \to \infty} L_{2k} = 4 \cdot 2 = 8 \).
1. (a) \( y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + \sec^4 x} \, dx \). By (7), an integral for the area of the surface obtained by rotating the curve about the \( x \)-axis is 
\[
S = \int 2\pi y \, ds = \int_0^{\pi/2} 2\pi \tan x \sqrt{1 + \sec^4 x} \, dx.
\]

(ii) By (8), an integral for the area of the surface obtained by rotating the curve about the \( y \)-axis is
\[
S = \int 2\pi x \, ds = \int_0^{\pi/2} 2\pi \tan x \sqrt{1 + \sec^4 x} \, dx.
\]

(b) (i) 10.5017 \quad (ii) 7.9353

2. (a) \( y = x^{-2} \Rightarrow \frac{dy}{dx} = -2x^{-3} \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + 4x^{-6}} \, dx \). By (7),
\[
S = \int 2\pi y \, ds = \int_1^\infty 2\pi x^{-2} \sqrt{1 + 4x^{-6}} \, dx.
\]

(ii) By (8), \( S = \int 2\pi x \, ds = \int_1^\infty 2\pi x \sqrt{1 + 4x^{-6}} \, dx \) \hspace{1cm} \text{[symmetric about the \( y \)-axis]}

(b) (i) 4.4566 \quad (ii) 11.7299

3. (a) \( y = e^{-x^2} \Rightarrow \frac{dy}{dx} = e^{-x^2} \cdot (-2x) \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + 4x^2e^{-2x^2}} \, dx \). By (7),
\[
S = \int 2\pi y \, ds = \int_{-1}^1 2\pi e^{-x^2} \sqrt{1 + 4x^2e^{-2x^2}} \, dx.
\]

(ii) By (8), \( S = \int 2\pi x \, ds = \int_0^1 2\pi x \sqrt{1 + 4x^2e^{-2x^2}} \, dx \)

(b) (i) 11.0753 \quad (ii) 3.9603

4. (a) \( x = \ln(2y + 1) \Rightarrow \frac{dx}{dy} = \frac{2}{2y + 1} \Rightarrow ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \sqrt{1 + \frac{4}{(2y + 1)^2}} \, dy \). By (7),
\[
S = \int 2\pi y \, ds = \int_0^1 2\pi y \sqrt{1 + \frac{4}{(2y + 1)^2}} \, dy.
\]

(ii) By (8), \( S = \int 2\pi x \, ds = \int_0^1 2\pi \ln(2y + 1) \sqrt{1 + \frac{4}{(2y + 1)^2}} \, dy \)

(b) (i) 4.2583 \quad (ii) 5.6053

5. \( y = x^2 \Rightarrow y' = 2x^3 \). So
\[
S = \int_0^y 2\pi y \sqrt{1 + (y')^2} \, dx = 2\pi \int_0^y x^2 \sqrt{1 + 9x^4} \, dx \quad [u = 1 + 9x^4, \, du = 36x^3 \, dx]
\]
\[
= \frac{2\pi}{36} \int_1^{145} \sqrt{u} \, du = \frac{\pi}{18} \left[ \frac{2u^{3/2}}{3} \right]_1^{145} = \frac{\pi}{27} (145\sqrt{145} - 1)
\]
6. The curve $9x = y^2 + 18$ is symmetric about the $x$-axis, so we only use its top half, given by $y = 3 \sqrt{x - 2}$.

\[ \frac{dy}{dx} = \frac{3}{2 \sqrt{x - 2}}, \text{ so } 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{9}{4(x - 2)}. \text{ Thus,} \]

\[ S = \int_2^6 2\pi \cdot 3 \sqrt{x - 2} \sqrt{1 + \frac{9}{4(x - 2)}} \, dx = 6\pi \int_2^6 \sqrt{x - 2 + \frac{9}{4}} \, dx = 6\pi \int_2^6 \left( x + \frac{1}{4} \right)^{1/2} \, dx \]

\[ = 6\pi \cdot \frac{2}{3} \left[ \left( x + \frac{1}{4} \right)^{3/2} \right]_2^6 = 4\pi \left[ \left( \frac{25}{4} \right)^{3/2} - \left( \frac{9}{4} \right)^{3/2} \right] = 4\pi \left( \frac{125}{8} - \frac{27}{8} \right) = 4\pi \frac{98}{8} = 49\pi \]

7. $y = \sqrt{1 + 4x} \Rightarrow y' = \frac{1}{2} (1 + 4x)^{-1/2} (4) = \frac{2}{\sqrt{1 + 4x}} \Rightarrow \sqrt{1 + (y')^2} = \sqrt{1 + \frac{4}{1 + 4x}} = \sqrt{\frac{5 + 4x}{1 + 4x}}$. So

\[ S = \int_0^5 2\pi y \sqrt{1 + (y')^2} \, dx = 2\pi \int_0^5 \sqrt{1 + 4x} \sqrt{\frac{5 + 4x}{1 + 4x}} \, dx = 2\pi \int_0^5 \sqrt{4x + 5} \, dx \]

\[ = 2\pi \int_0^5 \sqrt{u} \left( \frac{du}{4 dx} \right) = \frac{2\pi}{4} \left[ \frac{2}{3} u^{3/2} \right]_0^5 = \frac{2\pi}{3} \left( 25^{3/2} - 9^{3/2} \right) = \frac{2\pi}{3} (125 - 27) = \frac{98}{3} \pi \]

8. $y = \sqrt{1 + e^x} \Rightarrow y' = \frac{1}{2} (1 + e^x)^{-1/2} (e^x) = \frac{e^x}{2\sqrt{1 + e^x}} \Rightarrow$

\[ \sqrt{1 + (y')^2} = \sqrt{1 + \frac{e^{2x}}{4(1 + e^x)}} = \sqrt{\frac{4 + 4e^x + e^{2x}}{4(1 + e^x)}} = \frac{e^x + 2}{2\sqrt{1 + e^x}}. \text{ So} \]

\[ S = \int_0^1 2\pi y \sqrt{1 + (y')^2} \, dx = 2\pi \int_0^1 \sqrt{1 + e^x} \cdot \frac{e^x + 2}{2\sqrt{1 + e^x}} \, dx = \pi \int_0^1 (e^x + 2) \, dx \]

\[ = \pi [e^x + 2x]_0^1 = \pi [(e + 2) - (1 + 0)] = \pi (e + 1) \]

9. $y = \sin \pi x \Rightarrow y' = \pi \cos \pi x \Rightarrow 1 + (y')^2 = 1 + \pi^2 \cos^2(\pi x)$. So

\[ S = \int_0^1 2\pi y \sqrt{1 + (y')^2} \, dx = 2\pi \int_0^1 \sin \pi x \sqrt{1 + \pi^2 \cos^2(\pi x)} \, dx \]

\[ = 2\pi \int_0^\pi \sqrt{1 + u^2} \left( -\frac{1}{\pi^2} \, du \right) = \frac{2}{\pi} \int_0^\pi \sqrt{1 + u^2} \, du \]

\[ = \frac{4}{\pi} \left[ \frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln \left( u + \sqrt{1 + u^2} \right) \right]_0^\pi 
\]

\[ = \frac{4}{\pi} \left[ \left( \frac{\pi}{2} \sqrt{1 + \pi^2} + \frac{1}{2} \ln \left( \pi + \sqrt{1 + \pi^2} \right) \right) - 0 \right] = 2\sqrt{1 + \pi^2} + 2 \frac{\ln (\pi + \sqrt{1 + \pi^2})}{\pi} \]
10. \( y = \frac{x^2}{6} + \frac{1}{2x} \Rightarrow \frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} \Rightarrow \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{x^4}{4} + \frac{1}{1 + \frac{4}{2x^2}}} = \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \frac{x^2}{2} + \frac{1}{2x^2} \Rightarrow \\
S = \int_{1/2}^{1} 2\pi \left(\frac{x^2}{6} + \frac{1}{2x}\right) \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = 2\pi \int_{1/2}^{1} \left(\frac{x^5}{12} + \frac{x}{3} + \frac{x^{-3}}{4}\right) dx = 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{x^{-2}}{8}\right]_{1/2}^{1} \\
= 2\pi \left(\frac{1}{4} + \frac{1}{8} - \frac{1}{64} - \frac{1}{32} + \frac{1}{24} - \frac{1}{4}\right) = 2\pi \left(\frac{257}{256}\right) = \frac{257\pi}{256} \text{ (257/256)}.}

11. \( x = \frac{1}{2}(y^2 + 2)^{3/2} \Rightarrow \frac{dx}{dy} = \frac{1}{2}(y^2 + 2)^{1/2}(2y) = y \sqrt{y^2 + 2} \Rightarrow 1 + (\frac{dx}{dy})^2 = 1 + y^2(y^2 + 2) = (y^2 + 1)^2 \). 

So \( S = 2\pi \int_{1/2}^{1} y(y^2 + 1) dy = 2\pi \left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_{1/2}^{1} = 2\pi \left(2 - 1 - \frac{1}{8}\right) = \frac{2\pi}{8}. \)

12. \( x = 1 + 2y^2 \Rightarrow 1 + (\frac{dx}{dy})^2 = 1 + (4y)^2 = 1 + 16y^2. \)

So \( S = 2\pi \int_{0}^{\infty} \sqrt{1 + 16y^2} dy = \frac{\pi}{16} \int_{0}^{\infty} (16y^2 + 1)^{1/2} 32y dy = \frac{\pi}{16} \left[\frac{2}{3}(16y^2 + 1)^{3/2}\right]_{0}^{\infty} = \frac{\pi}{16} \left(65\sqrt{65} - 17\sqrt{17}\right). \)

13. \( y = \sqrt[3]{x} \Rightarrow x = y^3 \Rightarrow 1 + (\frac{dx}{dy})^2 = 1 + 9y^4. \) So 

\( S = 2\pi \int_{1}^{0} x \sqrt{1 + (\frac{dx}{dy})^2} dy = 2\pi \int_{1}^{0} y^2 \sqrt{1 + 9y^4} dy = \frac{\pi}{18} \int_{1}^{0} \sqrt[3]{1 + 9y^4} 36y^2 dy = \frac{\pi}{18} \left[\frac{3}{5}(1 + 9y^4)^{5/2}\right]_{1}^{0} = \frac{\pi}{18} \left(145\sqrt{145} - 10\sqrt{10}\right). \)

14. \( y = 1 - x^2 \Rightarrow 1 + (\frac{dx}{dy})^2 = 1 + 4x^2 \Rightarrow \\
S = 2\pi \int_{0}^{a} x \sqrt{1 + 4x^2} dx = \frac{\pi}{2} \int_{0}^{a} 8x \sqrt{4x^2 + 1} dx = \frac{\pi}{4} \left[\frac{2}{3}(4x^2 + 1)^{3/2}\right]_{0}^{a} = \frac{\pi}{6} \left(5\sqrt{5} - 1\right). \)

15. \( x = \sqrt{a^2 - y^2} \Rightarrow \frac{dx}{dy} = \frac{1}{2}(a^2 - y^2)^{-1/2}(-2y) = -y/\sqrt{a^2 - y^2} \Rightarrow \\
1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \Rightarrow \\
S = \int_{0}^{a/2} 2\pi \sqrt{a^2 - y^2} \cdot \frac{a}{\sqrt{a^2 - y^2}} dy = 2\pi \int_{0}^{a/2} a dy = 2\pi a \left[y\right]_{0}^{a/2} = 2\pi a \left(\frac{a}{2} - 0\right) = \pi a^2. \)

Note that this is \( \frac{1}{4} \) the surface area of a sphere of radius \( a \), and the length of the interval \( y = 0 \) to \( y = a/2 \) is \( \frac{1}{4} \) the length of the interval \( y = -a \) to \( y = a \).
16. If \( y = \frac{1}{4} x^2 - \frac{1}{3} \ln x \Rightarrow \frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x} \Rightarrow 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left( \frac{x}{2} + \frac{1}{2x} \right)^2 \), then

\[
S = \int_1^2 2\pi x \sqrt{\left( \frac{x}{2} + \frac{1}{2x} \right)^2} \, dx = 2\pi \int_1^2 x \left( \frac{x}{2} + \frac{1}{2x} \right) \, dx = \pi \int_1^2 (x^2 + 1) \, dx = \pi \left[ \frac{3}{2} x^2 + x \right]_1^2
\]

\[
= \pi \left[ \left( \frac{8}{2} + 2 \right) - \left( \frac{1}{2} + 1 \right) \right] = \frac{19}{2} \pi
\]

17. If \( y = x^3 \) and \( 0 \leq y \leq 1 \Rightarrow y' = 3x^2 \) and \( 0 \leq x \leq 1 \)

\[
S = \int_0^1 2\pi x \sqrt{1 + (3x^3)^2} \, dx = 2\pi \int_0^3 \sqrt{1 + u^2} \, \frac{du}{6} \quad \left[ u = 3x^3 \right]
\]

\[
= \frac{\pi}{3} \int_0^3 \sqrt{1 + u^2} \, du
\]

\[
\approx \frac{\pi}{3} \left[ \frac{1}{2} u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}) \right]_0^3 = \frac{\pi}{3} \left[ \frac{2}{2} \sqrt{10} + \frac{1}{2} \ln(3 + \sqrt{10}) \right] = \frac{\pi}{3} \left[ 3 \sqrt{10} + \ln(3 + \sqrt{10}) \right]
\]

18. If \( y = \ln(x + 1), 0 \leq x \leq 1 \). Then \( dS = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \sqrt{1 + \left( \frac{1}{x + 1} \right)^2} \, dx \), so

\[
S = \int_0^1 2\pi x \sqrt{1 + \left( \frac{1}{x + 1} \right)^2} \, dx = \int_1^2 2\pi (u - 1) \sqrt{1 + \frac{1}{u^2}} \, du \quad \left[ u = x + 1, \, du = dx \right]
\]

\[
= 2\pi \int_1^2 u \sqrt{1 + u^2} \, du - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u} \, du = 2\pi \int_1^2 \sqrt{1 + u^2} \, du - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u} \, du
\]

\[
\approx 2\pi \left[ \frac{1}{2} u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}) \right]_1^2 - 2\pi \left[ \sqrt{1 + u^2} - \ln \left( \frac{1 + \sqrt{1 + u^2}}{u} \right) \right]_1^2
\]

\[
= 2\pi \left[ \frac{1}{2} \ln(2 + \sqrt{5}) + \ln \left( \frac{1 + \sqrt{5}}{\sqrt{2}} \right) + \frac{\sqrt{2}}{2} - \frac{2}{2} \ln(1 + \sqrt{2}) \right]
\]
19. Since \( a > 0 \), the curve \( 3ay^2 = x(a - x)^2 \) only has points with \( x \geq 0 \).

\[ 3ay^2 \geq 0 \implies x(a - x)^2 \geq 0 \implies x \geq 0. \]

The curve is symmetric about the \( x \)-axis (since the equation is unchanged when \( y \) is replaced by \(-y\)). \( y = 0 \) when \( x = 0 \) or \( a \), so the curve’s loop extends from \( x = 0 \) to \( x = a \).

\[
\frac{d}{dx} (3ay^2) = \frac{d}{dx} [x(a - x)^2] = 6ay \frac{dy}{dx} = x \cdot 2(a - x)(-1) + (a - x)^2 \implies \frac{dy}{dx} = \frac{(a - x)[-2x + a - x]}{6ay} \implies
\]

\[
\left( \frac{dy}{dx} \right)^2 = \frac{(a - x)^2(a - 3x)^2}{36a^2y^2} = \frac{(a - x)^2(a - 3x)^2}{36a^2} \cdot \frac{3a}{x(a - x)^2} \quad \text{[the last fraction is } \frac{1}{y^2} \text{]}
\]

\[
1 + \left( \frac{dy}{dx} \right)^2 = \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{12ax}{12ax} + \frac{a^2 - 6ax + 9x^2}{12ax} = \frac{a^2 + 6ax + 9x^2}{12ax} = \frac{(a + 3x)^2}{12ax} \quad \text{for } x \neq 0.
\]

(a) \( S = \int_{x=0}^{a} 2\pi y \, ds = 2\pi \int_{0}^{a} \frac{\sqrt{x(a - x)}}{\sqrt{3a}} \cdot \frac{a + 3x}{\sqrt{12ax}} \, dx = 2\pi \int_{0}^{a} \frac{(a - x)(a + 3x)}{6a} \, dx
\]

\[
= \frac{\pi}{3a} \int_{0}^{a} \left( a^2 + 6x^2 - 3x^3 \right) \, dx = \frac{\pi}{3a} \left[ a^2 x + \frac{6}{5} x^5 - \frac{3}{4} x^4 \right]_{0}^{a} = \frac{\pi}{3a} \left( a^2 + \frac{6}{5} a^5 - \frac{3}{4} a^4 \right) = \frac{\pi}{3a} \cdot a^2 = \frac{\pi a^2}{3}.
\]

Note that we have rotated the top half of the loop about the \( x \)-axis. This generates the full surface.

(b) We must rotate the full loop about the \( y \)-axis, so we get double the area obtained by rotating the top half of the loop.

\[
S = 2 \cdot 2\pi \int_{x=0}^{a} x \, ds = 4\pi \int_{0}^{a} x \sqrt{12ax} \, dx = \frac{4\pi}{2\sqrt{3a}} \int_{0}^{a} x^{1/3} (a + 3x) \, dx = \frac{2\pi}{\sqrt{3a}} \int_{0}^{a} (ax^{1/3} + 3x^{2/3}) \, dx
\]

\[
= \frac{2\pi}{\sqrt{3a}} \left[ \frac{2}{3} a x^{2/3} + \frac{6}{5} x^{5/3} \right]_{0}^{a} = \frac{2\pi \sqrt{3}}{3 \sqrt{a}} \left( \frac{2}{3} a^{2/3} + \frac{6}{5} a^{5/3} \right) = \frac{2\pi \sqrt{3}}{3} \left( \frac{2}{3} + \frac{6}{5} \right) a^2 = \frac{2\pi \sqrt{3}}{3} \left( \frac{28}{15} \right) a^2
\]

\[
= \frac{56\pi \sqrt{3} a^2}{45}.
\]

20. In general, if the parabola \( y = ax^2, \quad -c \leq x \leq c \), is rotated about the \( y \)-axis, the surface area it generates is

\[
2\pi \int_{0}^{c} x \sqrt{1 + (2ax)^2} \, dx = 2\pi \int_{0}^{2ac} \frac{u}{2a} \sqrt{1 + u^2} \frac{1}{2a} \, du = \frac{\pi}{4a^2} \int_{0}^{2ac} (1 + u^2)^{1/2} 2u \, du
\]

\[
= \frac{\pi}{4a^2} \left[ \frac{2}{3} (1 + u^2)^{3/2} \right]_{0}^{2ac} = \frac{\pi}{6a^2} \left[ (1 + 4a^2c^2)^{3/2} - 1 \right]
\]

Here \( 2c = 10 \text{ ft} \) and \( ac^2 = 2 \text{ ft} \), so \( c = 5 \) and \( a = \frac{2}{25} \). Thus, the surface area is

\[
S = \frac{\pi}{6} \frac{625}{4} \left[ (1 + 4 \cdot \frac{2}{25} \cdot 25)^{3/2} - 1 \right] = \frac{625\pi}{24} \left[ (1 + 16 \cdot \frac{25}{25})^{3/2} - 1 \right] = \frac{625\pi}{24} \left( \frac{41}{125} - 1 \right) = \frac{5\pi}{24} (41 \sqrt{125} - 125) \approx 90.01 \text{ ft}^2.
\]
21. (a) \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y \, (dy/dx)}{b^2} = -\frac{x}{a^2} \Rightarrow \frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \Rightarrow \)

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{b^4 x^2}{a^4 y^2} = \frac{b^4 x^2 + a^4 y^2}{a^4 y^2} = \frac{b^4 x^2 + a^4 b^4 (1 - x^2/a^2)}{a^4 b^2 (1 - x^2/a^2)} = \frac{a^4 b^4 + b^4 x^2 - a^2 b^2 x^2}{a^4 b^2 - a^2 b^2 x^2} = \frac{a^4 + b^2 x^2 - a^2 x^2}{a^2 - x^2} \Rightarrow \frac{a^4}{a^2 - x^2} = \frac{a^4 - (a^2 - b^2) x^2}{a^2 (a^2 - x^2)}
\]

The ellipsoid’s surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the \( x \)-axis.

Thus,

\[
S = 2 \int_0^a 2 \pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 4 \pi \int_0^a b \, \sqrt{\frac{a^4 - x^2}{a^2 - x^2}} \, \sqrt{\frac{a^4 - (a^2 - b^2) x^2}{a^2 - x^2}} \, dx = \frac{4 \pi b}{a^2} \int_0^a \sqrt{\frac{a^4 - (a^2 - b^2) x^2}{a^2 - x^2}} \, dx
\]

\[
= \frac{4 \pi b}{a^2 \sqrt{a^2 - b^2}} \left[ \frac{2}{a^2} \sqrt{a^4 - a^2 (a^2 - b^2)} + \frac{a^4}{2} \sin^{-1} \left( \frac{a^2 - b^2}{a} \right) \right] = 2 \pi \left[ a^2 b \sin^{-1} \frac{a^2 - b^2}{a} \right]
\]

(b) \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x \, (dx/dy)}{a^2} = -\frac{y}{b^2} \Rightarrow \frac{dx}{dy} = -\frac{b^2}{a^2} y \Rightarrow \)

\[
1 + \left( \frac{dx}{dy} \right)^2 = 1 + \frac{b^4 y^2}{b^4 x^2} = \frac{b^4 y^2 + a^4 y^2}{b^4 x^2} = \frac{b^4 a^2 (1 - y^2/b^2) + a^4 y^2}{b^4 a^2 (1 - y^2/b^2)} = \frac{a^2 b^4 - a^2 b^2 y^2 + a^4 y^2}{a^2 b^4 - a^2 b^2 y^2} = \frac{b^4 - b^2 y^2 + a^2 y^2}{b^4 - b^2 y^2} = \frac{b^4 - (b^2 - a^2) y^2}{b^4 - b^2 y^2}
\]

The oblate spheroid’s surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the \( y \)-axis.

Thus,

\[
S = 2 \int_0^b 2 \pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = 4 \pi \int_0^b \frac{a}{b} \sqrt{b^2 - y^2} \, \frac{\sqrt{b^4 - (b^2 - a^2) y^2}}{b \sqrt{b^2 - y^2}} \, dy
\]

\[
= \frac{4 \pi a}{b^2} \int_0^b \sqrt{b^4 - (b^2 - a^2) y^2} \, dy = \frac{4 \pi a}{b^2} \int_0^b \sqrt{b^4 + (a^2 - b^2) y^2} \, dy = \frac{4 \pi a}{b^2} \int_0^b \frac{\sqrt{b^4 + u^2}}{\sqrt{a^2 - b^2}} \, du = \frac{21}{b^2 \sqrt{a^2 - b^2}} \left[ \frac{2}{b^2} \sqrt{b^4 + u^2} + \frac{b^4}{2} \ln \left( \frac{b^2 + b^2 y^2}{b^2 + u^2} \right) \right]_0
\]

\[
= \frac{4 \pi a}{b^2 \sqrt{a^2 - b^2}} \left\{ \left[ \frac{2}{b^2} \sqrt{b^4 + u^2} + \frac{b^4}{2} \ln \left( \frac{b^2 + b^2 y^2}{b^2 + u^2} \right) \right]_0 + \frac{4 \pi a}{b^2 \sqrt{a^2 - b^2}} \left[ \frac{a b^2 \sqrt{a^2 - b^2}}{2} + \frac{b^4}{2} \ln \left( \frac{b^2 + b^2 y^2}{b^2 + u^2} \right) \right]_0 \right\} = 2 \pi a^2 + \frac{2 \pi a b^2}{\sqrt{a^2 - b^2}} \ln \left( \frac{a^2 - b^2}{b} \right)
\]
22. The upper half of the torus is generated by rotating the curve $(x - R)^2 + y^2 = r^2$, $y > 0$, about the $y$-axis.

\[
y' \frac{dy}{dx} = -(x - R) = 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \frac{x - R}{y} \right)^2 = \frac{y^2 + (x - R)^2}{y^2} = \frac{r^2}{r^2 - (x - R)^2}
\]

Thus,

\[
S = 2 \int_{R-r}^{R+r} 2\pi x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 4\pi \int_{R-r}^{R+r} \frac{r x}{\sqrt{r^2 - (x - R)^2}} \, dx = 4\pi r \int_{R-r}^{R+r} \frac{u + R}{\sqrt{r^2 - u^2}} \, du
\]

\[
= 4\pi r \int_{r}^{r} \frac{u \, du}{\sqrt{r^2 - u^2}} + 4\pi r R \int_{r}^{r} \frac{du}{\sqrt{r^2 - u^2}} = 4\pi r \cdot 0 + 8\pi R r \int_{0}^{r} \frac{du}{\sqrt{r^2 - u^2}} \quad \text{[since the first integrand is odd and the second is even]}
\]

\[
= 8\pi R r \left[ \sin^{-1} \left( \frac{u}{r} \right) \right]_0^r = 8\pi R r \left( \frac{\pi}{2} \right) = 4\pi^2 R r
\]

23. The analogue of $f(x_i^*)$ in the derivation of (4) is now $c - f(x_i^*)$, so

\[
S = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \left[ c - f(x_i^*) \right] \sqrt{1 + \left[ f'(x_i^*) \right]^2} \Delta x = \int_a^b 2\pi [c - f(x)] \sqrt{1 + \left[ f'(x) \right]^2} \, dx.
\]

24. $y = x^{1/2} \Rightarrow y' = \frac{1}{2} x^{-1/2} \Rightarrow 1 + (y')^2 = 1 + 1/4x$, so by Exercise 23, $S = \int_a^b 2\pi \left( 4 - \sqrt{x} \right) \sqrt{1 + 1/(4x)} \, dx$.

Using a CAS, we get $S = 2\pi \ln \left( \sqrt{17} + 4 \right) + \frac{\pi}{8} \left( 31 \sqrt{17} + 1 \right) \approx 80.6905$.

25. For the upper semicircle, $f(x) = \sqrt{r^2 - x^2}$, $f'(x) = -x/\sqrt{r^2 - x^2}$. The surface area generated is

\[
S_1 = \int_{-r}^{r} 2\pi \left( r - \sqrt{r^2 - x^2} \right) \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx = 4\pi \int_{0}^{r} \left( r - \sqrt{r^2 - x^2} \right) \frac{r}{\sqrt{r^2 - x^2}} \, dx
\]

\[
= 4\pi \int_{0}^{r} \left( \frac{r^2}{\sqrt{r^2 - x^2}} - r \right) \, dx
\]

For the lower semicircle, $f(x) = -\sqrt{r^2 - x^2}$ and $f'(x) = \frac{x}{\sqrt{r^2 - x^2}}$, so $S_2 = 4\pi \int_{0}^{r} \left( \frac{r^2}{\sqrt{r^2 - x^2}} + r \right) \, dx$.

Thus, the total area is $S = S_1 + S_2 = 8\pi \int_{0}^{r} \left( \frac{r^2}{\sqrt{r^2 - x^2}} \right) \, dx = 8\pi \left[ r^2 \sin^{-1} \left( \frac{x}{r} \right) \right]_0^r = 8\pi r^2 \left( \frac{\pi}{2} \right) = 4\pi^2 r^2$.

26. (a) Rotate $y = \sqrt{R^2 - x^2}$ with $a \leq x \leq a + h$ about the $x$-axis to generate a zone of a sphere. $y = \sqrt{R^2 - x^2} \Rightarrow

\[
y' = \frac{1}{2} (R^2 - x^2)^{-1/2} (-2x) \Rightarrow ds = \sqrt{1 + \left( \frac{-x}{\sqrt{R^2 - x^2}} \right)^2} \, dx.
\]

The surface area is

\[
S = \int_{a}^{a+h} 2\pi y \, ds = 2\pi \int_{a}^{a+h} \sqrt{R^2 - x^2} \sqrt{1 + \frac{x^2}{R^2 - x^2}} \, dx
\]

\[
= 2\pi \int_{a}^{a+h} \sqrt{R^2 - x^2} + x^2 \, dx = 2\pi R \left[ x \right]_a^{a+h}
\]

\[
= 2\pi R (a + h - a) = 2\pi Rh
\]

(b) Rotate $y = R$ with $0 \leq x \leq h$ about the $x$-axis to generate a zone of a cylinder. $y = R \Rightarrow y' = 0 \Rightarrow

\[
ds = \sqrt{1 + 0^2} \, dx = dx.
\]

The surface area is $S = \int_{0}^{h} 2\pi y \, ds = 2\pi \int_{0}^{h} R \, dx = 2\pi R \left[ x \right]_0^{h} = 2\pi Rh$. 
1. \[ W = \int_{x_{0}}^{x_{f}} f(x) \, dx = \int_{1}^{10} 5x^{-2} \, dx = 5\left[-x^{-1}\right]_{1}^{10} = 5\left(-\frac{1}{10} + 1\right) = 4.5 \text{ ft-lb} \]

2. \[ W = \int_{1}^{2} \cos \left(\frac{1}{2} \pi x \right) \, dx = \frac{2}{\pi} \left[ \sin \left(\frac{1}{2} \pi x \right) \right]_{1}^{2} = \frac{2}{\pi} \left( \frac{\sqrt{2}}{2} - \frac{-\sqrt{2}}{2} \right) = 0 \text{ Nm} = 0 J \]

Interpretation: From \( x = 1 \) to \( x = \frac{3}{2} \), the force does work equal to \( \int_{1}^{3/2} \cos \left(\frac{1}{2} \pi x \right) \, dx \), which accelerates the particle and increases its kinetic energy. From \( x = \frac{3}{2} \) to \( x = 2 \), the force opposes the motion of the particle, decreasing its kinetic energy. This is negative work, equal in magnitude but opposite in sign to the work done from \( x = 1 \) to \( x = \frac{3}{2} \).

3. The force function is given by \( F(x) \) (in newtons) and the work (in joules) is the area under the curve, given by
\[ \int_{0}^{8} F(x) \, dx = \int_{0}^{4} F(x) \, dx + \int_{4}^{8} F(x) \, dx = \frac{1}{2}(4)(30) + (4)(30) = 180 J \]

4. Work = \[ \int_{0}^{15} f(x) \, dx \approx S \approx \int_{0}^{18} \left[ f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18) \right] \]
\[ = 1 \cdot \left[ 9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4 \right] = 148 \text{ joules} \]

5. According to Hooke’s Law, the force required to maintain a spring stretched \( x \) units beyond its natural length is proportional to \( x \), that is, \( f(x) = kx \). Here, the amount stretched is \( x = 4 \text{ in.} \) = \( \frac{1}{2} \text{ ft} \) and the force is 10 lb. Thus, \( 10 = k \left( \frac{1}{2} \right) \Rightarrow k = 30 \text{ lb/ft} \), and \( f(x) = 30x \). The work done in stretching the spring from its natural length to \( 6 \text{ in.} \) = \( \frac{1}{2} \text{ ft} \) beyond its natural length is
\[ W = \int_{0}^{1/2} 30x \, dx = \left[ 15x^2 \right]_{0}^{1/2} = \frac{15}{4} \text{ ft-lb} \]

6. According to Hooke’s Law, the force required to maintain a spring stretched \( x \) units beyond its natural length is proportional to \( x \), that is, \( f(x) = kx \). Here, the amount stretched is \( x = 30 - 20 = 10 \text{ cm} = 0.1 \text{ m} \) and the force is 25 N. Thus, \( 25 = k(0.1) \Rightarrow k = 250 \text{ N/m} \), and \( f(x) = 250x \). The work required to stretch the spring from 20 cm to 25 cm
\[ [25 - 20 = 5 \text{ cm} = 0.05 \text{ m}] \text{ is } W = \int_{0}^{0.05} 250x \, dx = \left[ 125x^2 \right]_{0}^{0.05} = 125(0.0025) = 0.3125 \approx 0.31 J \]

7. (a) If \( \int_{0}^{0.12} kx \, dx = 2 \text{ J} \), then \( 2 = \left[ \frac{1}{2} kx^2 \right]_{0}^{0.12} = \frac{1}{2} k(0.0144) = 0.0072k \) and \( k = \frac{2}{0.0072} = \frac{2800}{9} \approx 277.8 \text{ N/m} \).

Thus, the work needed to stretch the spring from 35 cm to 40 cm is
\[ \int_{0.05}^{0.10} \frac{2600}{9} x \, dx = \left[ \frac{1250x^2}{9} \right]_{1/40}^{1/20} = \frac{1250}{9} \left( \frac{1}{100} - \frac{1}{400} \right) = \frac{25}{24} \approx 1.04 \text{ J} \]

(b) \( f(x) = kx \), so \( 30 = \frac{2600}{9} x \) and \( x = \frac{270}{2300} \text{ m} = 10.8 \text{ cm} \)

8. Let \( L \) be the natural length of the spring in meters. Then
\[ 6 = \int_{0}^{0.12-L} kx \, dx = \left[ \frac{1}{2} kx^2 \right]_{0}^{0.12-L} = \frac{1}{2} k \left[ (0.12 - L)^2 - (0.10 - L)^2 \right] \]

and
\[ 10 = \int_{0.12-L}^{0.14-L} kx \, dx = \left[ \frac{1}{2} kx^2 \right]_{0.12-L}^{0.14-L} = \frac{1}{2} k \left[ (0.14 - L)^2 - (0.12 - L)^2 \right] \]

Simplifying gives us \( 12 = k(0.0044 - 0.04L) \) and \( 20 = k(0.0052 - 0.04L) \). Subtracting the first equation from the second gives \( 8 = 0.0008k \), so \( k = 10,000 \). Now the second equation becomes \( 20 = 52 - 400L \), so \( L = \frac{28}{400} \text{ m} = 8 \text{ cm} \).
In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x^*_i \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

9. (a) The portion of the rope from \( x \) ft to \((x + \Delta x)\) ft below the top of the building weighs \( \frac{1}{2} \Delta x \) lb and must be lifted \( x^*_i \) ft, so its contribution to the total work is \( \frac{1}{2} x^*_i \Delta x \) ft-lb. The total work is

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} x^*_i \Delta x = \int_0^{50} \frac{1}{2} x \, dx = \left[ \frac{1}{4} x^2 \right]_0^{50} = \frac{2500}{4} = 625 \text{ ft-lb}
\]

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

(b) When half the rope is pulled to the top of the building, the work to lift the top half of the rope is

\[
W_1 = \int_0^{25} \frac{1}{2} x \, dx = \left[ \frac{1}{4} x^2 \right]_0^{25} = \frac{625}{4} \text{ ft-lb}
\]

The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is

\[
W_2 = \int_0^{50} \frac{1}{2} \cdot 25 \, dx = \frac{25}{2} \left[ x \right]_0^{50} = \frac{625}{2} \text{ ft-lb}
\]

The total work done in pulling half the rope to the top of the building is

\[
W = W_1 + W_2 = \frac{625}{2} + \frac{625}{4} = \frac{3}{4} \cdot 625 = \frac{1875}{4} \text{ ft-lb}
\]

In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x^*_i \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

10. Assumptions:

1. After lifting, the chain is L-shaped, with 4 m of the chain lying along the ground.
2. The chain slides effortlessly and without friction along the ground while its end is lifted.
3. The weight density of the chain is constant throughout its length and therefore equals \((8 \text{ kg/m})(9.8 \text{ m/s}^2) = 78.4 \text{ N/m}\).

The part of the chain \( x \) m from the lifted end is raised \( \delta - x \) m if \( 0 \leq x \leq \delta \) m, and it is lifted 0 m if \( x > \delta \) m. Thus, the work needed is

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} (\delta - x^*_i) \cdot 78.4 \Delta x = \int_0^{\delta} (\delta - x)78.4 \, dx = 78.4 \left[ \delta x - \frac{1}{2} x^2 \right]_0^{\delta} = (78.4)(18) = 1411.2 \text{ J}
\]

In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x^*_i \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

11. The work needed to lift the cable is

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} 2x^*_i \Delta x = \int_0^{500} 2x \, dx = \left[ x^2 \right]_0^{500} = 250,000 \text{ ft-lb}
\]

The work needed to lift the coal is 800 lb \cdot 500 ft = 400,000 ft-lb. Thus, the total work required is 250,000 + 400,000 = 650,000 ft-lb.

In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x^*_i \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

12. The work needed to lift the bucket itself is 4 lb \cdot 80 ft = 320 ft-lb. At time \( t \) (in seconds) the bucket is \( x^*_t = 2 \) ft above its original 80 ft depth, but it now holds only \((40 - 0.2t)\) lb of water. In terms of distance, the bucket holds \([40 - 0.2(\frac{1}{2} x^*_t)]\) lb of water when it is \( x^*_t \) ft above its original 80 ft depth. Moving this amount of water a distance \( \Delta x \) requires \((40 - \frac{1}{10} x^*_t) \Delta x \) ft-lb of work. Thus, the work needed to lift the water is

\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} (40 - \frac{1}{10} x^*_i) \Delta x = \int_0^{80} (40 - \frac{1}{10} x) \, dx = \left[ 40x - \frac{1}{20} x^2 \right]_0^{80} = (3200 - 320) \text{ ft-lb}
\]

Adding the work of lifting the bucket gives a total of 3200 ft-lb of work.
In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x_i^* \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

13. At a height of \( x \) meters (\( 0 \leq x \leq 12 \)), the mass of the rope is \((0.8 \, \text{kg/m}) (12 - x \, \text{m}) = (9.6 - 0.8x) \, \text{kg}\) and the mass of the water is \((\frac{28}{9} \, \text{kg/m}) (12 - x \, \text{m}) = (36 - 3x) \, \text{kg}\). The mass of the bucket is 10 kg, so the total mass is
\[(9.6 - 0.8x) + (36 - 3x) + 10 = (55.6 - 3.8x) \, \text{kg},\]and hence, the total force is \(9.8(55.6 - 3.8x)\) N. The work needed to lift the bucket \( \Delta x \) m through the \( i \)th subinterval of \([0, 12]\) is \(9.8(55.6 - 3.8x_i^*)\) \( \Delta x \), so the total work is
\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} 9.8(55.6 - 3.8x_i^*) \Delta x = \int_{0}^{12} (9.8)(55.6 - 3.8x) \, dx = 9.8 \left[ 55.6x - 1.9x^2 \right]_{0}^{12} = 9.8(393.6) \approx 3857 \, \text{J}
\]

In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x_i^* \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

14. The chain’s weight density is \(\frac{25 \, \text{lb}}{10 \, \text{ft}} = 2.5 \, \text{lb/ft}\). The part of the chain \( x \) ft below the ceiling (for \( 5 \leq x \leq 10 \)) has to be lifted 2 \((x - 5)\) ft, so the work needed to lift the \( i \)th subinterval of the chain is \(2(x_i^* - 5)(2.5 \Delta x)\). The total work needed is
\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} 2(x_i^* - 5)(2.5) \Delta x = \int_{5}^{10} [2(x - 5)(2.5)] \, dx = 5 \int_{5}^{10} (x - 5) \, dx
\]
\[
= 5\left[ \frac{1}{2}x^2 - 5x \right]_{5}^{10} = 5\left[ (50 - 50) - (\frac{25}{2} - 25) \right] = 5\left( \frac{25}{2} \right) = 62.5 \, \text{ft-lb}
\]

In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x_i^* \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

15. A “slice” of water \( \Delta x \) m thick and lying at a depth of \( x_i^* \) m (where \( 0 \leq x_i^* \leq \frac{1}{2} \)) has volume \((2 \times 1 \times \Delta x) \, \text{m}^3\), a mass of \(2000 \Delta x \, \text{kg}\), weighs about \((9.8)(2000 \Delta x) = 19600 \Delta x \, \text{N}\), and thus requires about \(19600x_i^* \Delta x \, \text{J}\) of work for its removal.

So \( W = \lim_{n \to \infty} \sum_{i=1}^{n} 19600x_i^* \Delta x = \int_{0}^{1/2} 19600x \, dx = \left[ 9800x^2 \right]_{0}^{1/2} = 2450 \, \text{J}.\)

In Exercises 9–16, \( n \) is the number of subintervals of length \( \Delta x \), and \( x_i^* \) is a sample point in the \( i \)th subinterval \([x_{i-1}, x_i]\).

16. A horizontal cylindrical slice of water \( \Delta x \) m thick has a volume of \( \pi r^2 h = \pi \cdot 5^2 \cdot \Delta x \, \text{m}^3\) and weighs about \((1000 \, \text{kg/m}^3)(25 \pi \Delta x \, \text{m}^3) = 25000 \pi \Delta x \, \text{kg}\). If the slice lies \( x_i^* \) m below the edge of the pool (where \( 0.3 \leq x_i^* \leq 1.5 \)), then the work needed to pump it out is about \(25000 \pi x_i^* \Delta x\). Thus,
\[
W = \lim_{n \to \infty} \sum_{i=1}^{n} 25000 \pi x_i^* \Delta x = \int_{0.3}^{1.5} 25000 \pi x \, dx = \left[ 12500 \pi x^2 \right]_{0.3}^{1.5} = 12500 \pi (2.25 - 0.09) = 27000 \pi \, \text{J}
\]
17. (a) A rectangular “slice” of water $\Delta x$ m thick and lying $x$ m above the bottom has width $x$ m and volume $8x \Delta x$ m$^2$.

It weighs about $(9.8 \times 1000) (8x \Delta x)$ N, and must be lifted $(5 - x)$ m by the pump, so the work needed is about $(9.8 \times 10^2)(5 - x)(8x \Delta x)$ J. The total work required is

$$W \approx \int_0^3 (9.8 \times 10^2)(5 - x)8x \, dx = (9.8 \times 10^2) \int_0^3 (40x - 8x^2) \, dx = (9.8 \times 10^2) \left[20x^2 - \frac{8}{3}x^3\right]_0^3$$

$$= (9.8 \times 10^2)(180 - 72) = (9.8 \times 10^2)(108) = 1058.4 \times 10^2 \approx 1.06 \times 10^5 \text{ J}$$

(b) If only $4.7 \times 10^5$ J of work is done, then only the water above a certain level (call it $h$) will be pumped out. So we use the same formula as in part (a), except that the work is fixed, and we are trying to find the lower limit of integration:

$$4.7 \times 10^5 \approx \int_h^3 (9.8 \times 10^2)(5 - x)8x \, dx = (9.8 \times 10^2) \left[20x^2 - \frac{8}{3}x^3\right]_h^3 \Rightarrow$$

$$\frac{4.7}{0.8} \times 10^2 \approx 48 = (20 \cdot 3^2 - \frac{8}{3} \cdot 3^3) - (20h^2 - \frac{8}{3}h^3) \Rightarrow$$

$$2h^2 - 15h^2 + 45 = 0.$$ To find the solution of this equation, we plot $2h^2 - 15h^2 + 45$ between $h = 0$ and $h = 3$.

We see that the equation is satisfied for $h \approx 2.0$. So the depth of water remaining in the tank is about 2.0 m.

18. (a) Let $y$ measure depth (in meters) below the center of the spherical tank, so that $y = -3$ at the top of the tank and $y = -4$ at the spigot. A horizontal disk-shaped “slice” of water $\Delta y$ m thick and lying at coordinate $y$ has radius $\sqrt{9 - y^2}$ m and volume $\pi r^2 \Delta y = \pi (9 - y^2) \Delta y$ m$^3$. It weighs about $(9.8 \times 1000)\pi (9 - y^2) \Delta y$ N and must be lifted $(y + 4)$ m by the pump, so the work needed to pump it out is about $(9.8 \times 10^2)(y + 4)\pi (9 - y^2) \Delta y$ J. The total work required is

$$W \approx \int_{-3}^0 (9.8 \times 10^2)(y + 4)\pi (9 - y^2) \, dy = (9.8 \times 10^2)\pi \int_{-3}^0 (9y - y^3 + 36 - 4y^2) \, dy$$

$$= (9.8 \times 10^2)\pi (2)\int_0^9 (9y - y^3) \, dy \quad [\text{by Theorem 5.5.7}]$$

$$= (78.4 \times 10^2)\pi \left[9y - \frac{1}{2}y^3\right]_0^9 = (78.4 \times 10^2)\pi (18) = 1,411,200\pi \approx 4.43 \times 10^6 \text{ J}$$

(b) The only changes needed in the solution for part (a) are: (1) change the lower limit from $-3$ to $0$ and (2) change 1000 to 900.

$$W \approx \int_0^9 (9.8 \times 900)(y + 4)\pi (9 - y^2) \, dy = (9.8 \times 900)\pi \int_0^9 (9y - y^3 + 36 - 4y^2) \, dy$$

$$= (9.8 \times 900)\pi \left[\frac{3}{2}y^2 - \frac{1}{2}y^4 + 36y - \frac{8}{3}y^3\right]_0^9 = (9.8 \times 900)\pi (92.25) = 813,645\pi$$

$$\approx 2.56 \times 10^6 \text{ J} \quad [\text{about 58% of the work in part (a)}]$$

19. $V = \pi r^2 x$, so $V$ is a function of $x$ and $P$ can also be regarded as a function of $x$. If $V_1 = \pi r^2 x_1$ and $V_2 = \pi r^2 x_2$, then

$$W = \int_{x_1}^{x_2} F(x) \, dx = \int_{x_1}^{x_2} \pi r^2 P(V(x)) \, dx = \int_{x_1}^{x_2} P(V(x)) \, dV(x) \quad [\text{Let } V(x) = \pi r^2 x, \text{ so } dV(x) = \pi r^2 \, dx.]$$

$$= \int_{V_1}^{V_2} P(V) \, dV \quad \text{by the Substitution Rule.}$$
20. 160 lb/in² = 160 \cdot 144 \text{ lb/ft²}, 100 \text{ in²} = \frac{100}{1728} \text{ ft²}, \text{ and } 800 \text{ in}² = \frac{800}{1728} \text{ ft}². \vspace{.5em}

k = PV^{1.4} = (160 \cdot 144)\left(\frac{100}{1728}\right)^{1.4} = 23,040 \left(\frac{25}{432}\right)^{1.4} \approx 426.5. \text{ Therefore, } P \approx 426.5V^{-1.4} \text{ and} \vspace{.5em}

W = \int_{\frac{100}{1728}}^{\frac{800}{1728}} 426.5V^{-1.4} \, dV = 426.5 \left[-\frac{1}{-0.4}V^{-0.4}\right]_{\frac{100}{1728}}^{\frac{800}{1728}} = (426.5)(2.5) \left(\left(\frac{425}{25}\right)^{0.4} - \left(\frac{25}{25}\right)^{0.4}\right) \approx 1.88 \times 10^9 \text{ ft-lb.} \vspace{1cm}

21. (a) \[ W = \int_{a}^{b} F(r) \, dr = \int_{a}^{b} \frac{Gm_1m_2}{r^2} \, dr = Gm_1m_2 \left[\frac{-1}{r}\right]_{a}^{b} = Gm_1m_2 \left(\frac{1}{a} - \frac{1}{b}\right) \vspace{.5em} \]

(b) By part (a), \[ W = GMm\left(\frac{1}{R} - \frac{1}{R + 1,000,000}\right) \] where \( M \) = mass of the earth in kg, \( R \) = radius of the earth in m, and \( m \) = mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,

\[ W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1000) \times \left(\frac{1}{6.37 \times 10^6} - \frac{1}{7.37 \times 10^6}\right) \approx 8.50 \times 10^9 \text{ J} \vspace{1cm}

22. \[ W = \int_{R}^{\infty} \frac{GMm}{r^2} \, dr = \lim_{t \to \infty} \int_{R}^{t} \frac{GMm}{r^2} \, dr = \lim_{t \to \infty} GMm \left[\frac{-1}{r}\right]_{R}^{t} = GMm \lim_{t \to \infty} \left(\frac{-1}{t} + \frac{1}{R}\right) = \frac{GMm}{R}, \] where \( M \) = mass of the earth = 5.98 \times 10^{24} \text{ kg}, \( m \) = mass of the satellite = 10^3 \text{ kg,} \( R \) = radius of the earth = 6.37 \times 10^6 \text{ m,} \text{ and } G = \text{ gravitational constant} = 6.67 \times 10^{-11} \text{ N-m}^2/\text{kg}^2. \vspace{.5em}

Therefore, work = \[ \frac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24} \cdot 10^3}{6.37 \times 10^6} \approx 6.26 \times 10^{10} \text{ J.} \vspace{1cm}

23. From Exercise 22, \[ W = \frac{GMm}{R}. \] The initial kinetic energy supplies the needed work, so \[ \frac{1}{2}mv_0^2 = \frac{GMm}{R} \Rightarrow \] \[ v_0 = \sqrt{2GM/R}. \]
24. (a) Assume the pyramid has smooth sides. From the figure for 
\[
0 \leq x \leq 378, \text{ an equation for the side is } y = \frac{481}{378} x + 481 \quad \Rightarrow \\
\quad x = -\frac{278}{481} (y - 481). \text{ The horizontal length of a cross-section is} \\
\quad 2x \text{ and the area of a cross-section is} \\
\quad A = (2x)^2 = 4x^2 = \frac{378^2}{481^2} (y - 481)^2. \text{ A slice of thickness} \\
\quad \Delta y \text{ at height } y \text{ has volume } \Delta V = A \Delta y \text{ ft}^3 \text{ and weight} \\
\quad 150 \Delta V \text{ lb}, \text{ so the work needed to build the pyramid was} \\
\quad \begin{align*}
W_1 &= \int_{0}^{481} 150y \cdot 4 \frac{378^2}{481^2} (y - 481)^2 \, dy = 600 \frac{378^2}{481^2} \int_{0}^{481} (y^3 - 2 \cdot 481y^2 + 481^2y) \, dy \\
&= 600 \frac{378^2}{481^2} \left[ \frac{1}{4} y^4 - \frac{2}{3} \cdot 481y^3 + \frac{481^2}{2}y^2 \right]_0^{481} = 600 \frac{378^2}{481^2} \left( \frac{481^4}{4} - \frac{2 \cdot 481^4}{3} + \frac{481^4}{2} \right) \\
&= 600 \frac{378^2}{481^2} \frac{481^4}{12} = 50 \cdot 378^2 \cdot 481^2 \approx 1.653 \times 10^{12} \text{ ft-lb} \\
\end{align*}
\]
(b) Work done \( W_2 = 10 \text{ h/day} \cdot 340 \text{ days/year} \cdot 20 \text{ yr/1 laborer} \cdot 200 \text{ ft-lb/hour} = 1.36 \times 10^7 \text{ ft-lb/laborer} \) Dividing \( W_1 \) by \( W_2 \) gives us about 121,536 laborers.

25. Set up a vertical \( x \)-axis as shown, with \( x = 0 \) at the water’s surface and \( x \) increasing in the downward direction. Then the area of the \( i \)-th rectangular strip is \( 6 \Delta x \) and the pressure on the strip is \( \delta x_i^+ \) (where \( \delta \approx 62.5 \text{ lb/ft}^3 \)). Thus, the hydrostatic force on the strip is \( \delta x_i^+ \cdot 6 \Delta x \) and the total hydrostatic force \( \approx \sum_{i=1}^{n} \delta x_i^+ \cdot 6 \Delta x \). The total force
\[
F = \lim_{n \to \infty} \sum_{i=1}^{n} \delta x_i^+ \cdot 6 \Delta x = \int_{2}^{6} \delta x \cdot 6 \, dx = 6\delta \int_{2}^{6} x \, dx = 6\delta \left[ \frac{1}{2} x^2 \right]_{2}^{6} = 6\delta (18 - 2) = 96\delta \approx 6000 \text{ lb}.
\]

26. Set up a vertical \( x \)-axis as shown. Then the area of the \( i \)-th rectangular strip is
\[
\frac{4}{3} (4 - x_i^+) \Delta x. \quad \text{[By similar triangles, } \frac{w_i}{4 - x_i^+} = \frac{4}{3}, \text{ so } w_i = \frac{4}{3} (4 - x_i^+).]\]
The pressure on the strip is \( \delta x_i^+ \), so the hydrostatic force on the strip is
\[
\delta x_i^+ \cdot \frac{4}{3} (4 - x_i^+) \Delta x \text{ and the total force on the plate } \approx \sum_{i=1}^{n} \delta x_i^+ \cdot \frac{4}{3} (4 - x_i^+) \Delta x. \text{ The total force} \\
F = \lim_{n \to \infty} \sum_{i=1}^{n} \delta x_i^+ \cdot \frac{4}{3} (4 - x_i^+) \Delta x = \int_{1}^{4} \delta x \cdot \frac{4}{3} (4 - x) \, dx = \frac{4}{3} \delta \int_{1}^{4} (4x - x^2) \, dx \\
= \frac{4}{3} \delta \left[ 2x^2 - \frac{1}{2} x^3 \right]_{1}^{4} = \frac{4}{3} \delta \left[ (32 - \frac{64}{3}) - (2 - \frac{1}{2}) \right] = \frac{8}{3} \delta (9) = 12\delta \approx 750 \text{ lb}.
\]
27. Set up a vertical x-axis as shown. The base of the triangle shown in the figure has length \( \sqrt{3^2 - (x_i^*)^2} \), so \( w_i = 2 \sqrt{9 - (x_i^*)^2} \), and the area of the ith rectangular strip is \( 2 \sqrt{9 - (x_i^*)^2} \Delta x \). The ith rectangular strip is \( (x_i^* - 1) \) m below the surface level of the water, so the pressure on the strip is \( \rho g (x_i^* - 1) \).

The hydrostatic force on the strip is \( \rho g (x_i^* - 1) \cdot 2 \sqrt{9 - (x_i^*)^2} \Delta x \) and the total force on the plate \( \approx \sum_{i=1}^{n} \rho g (x_i^* - 1) \cdot 2 \sqrt{9 - (x_i^*)^2} \Delta x \). The total force

\[
F = \lim_{n \to \infty} \sum_{i=1}^{n} \rho g (x_i^* - 1) \cdot 2 \sqrt{9 - (x_i^*)^2} \Delta x = 2 \rho g \int_{1}^{3} (x - 1) \sqrt{9 - x^2} \, dx
\]

\[
= 2 \rho g \left[ -\frac{1}{3} (9 - x^2)^{3/2} \right]_{1}^{3} - 2 \rho g \left[ \frac{x}{2} \sqrt{9 - x^2} + \frac{9}{2} \sin^{-1} \left( \frac{x}{3} \right) \right]_{1}^{3}
\]

\[
= 2 \rho g \left[ 0 + \frac{3}{8} \left( 8 \sqrt{8} \right) - (\frac{1}{2} \sqrt{9} + \frac{9}{2} \sin^{-1} \left( \frac{3}{3} \right)) \right]
\]

\[
= \frac{2}{3} \sqrt{2} \rho g - \frac{9}{4} \rho g + 2 \sqrt{2} \rho g + 9 \sin^{-1} \left( \frac{1}{3} \right) \rho g = \left( \frac{28}{3} \sqrt{2} - \frac{9}{4} + 9 \sin^{-1} \left( \frac{1}{3} \right) \right) \rho g
\]

\[
\approx 6.835 \cdot 1000 \cdot 9.8 \approx 6.7 \times 10^4 \text{ N}
\]

Note: If you set up a typical coordinate system with the water level at \( y = -1 \), then \( F = \int_{-3}^{3} \rho g (-1 - y) 2 \sqrt{9 - y^2} \, dy \).

28. By similar triangles, \( w_i / 4 = x_i^* / 4 \), so \( w_i = \frac{3}{4} x_i^* \) and the area of the ith strip is \( \frac{3}{4} x_i^* \Delta x \).

The pressure on the strip is \( \rho g x_i^* \), so the hydrostatic force on the strip is \( \rho g x_i^* \cdot \frac{3}{4} x_i^* \Delta x \) and the total force on the plate \( \approx \sum_{i=1}^{n} \rho g x_i^* \cdot \frac{3}{4} x_i^* \Delta x \). The total force

\[
F = \lim_{n \to \infty} \sum_{i=1}^{n} \rho g x_i^* \cdot \frac{3}{4} x_i^* \Delta x = \int_{0}^{5} \rho g x \cdot \frac{3}{4} x \, dx = \frac{3}{4} \rho g \left[ \frac{1}{3} x^3 \right]_{0}^{5} = \frac{3}{4} \rho g \cdot \frac{125}{3} = \frac{100}{3} \rho g
\]

\[
\approx \frac{100}{3} \cdot 1000 \cdot 9.8 \approx 3.3 \times 10^5 \text{ N}
\]
29. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is
\[
\left(2 - \frac{2}{\sqrt{3}} x_i^*\right) \Delta x. \quad \text{[By similar triangles, } \frac{w_i}{2} = \frac{\sqrt{3} - x_i^*}{\sqrt{3}}, \text{ so } w_i = 2 - \frac{2}{\sqrt{3}} x_i^*\text{]}
\]

The pressure on the strip is \(\rho g x_i^*\), so the hydrostatic force on the strip is
\[
\rho g x_i^* \left(2 - \frac{2}{\sqrt{3}} x_i^*\right) \Delta x \text{ and the hydrostatic force on the plate } \approx \sum_{i=1}^{n} \rho g x_i^* \left(2 - \frac{2}{\sqrt{3}} x_i^*\right) \Delta x.
\]

The total force
\[
F = \lim_{n \to \infty} \sum_{i=1}^{n} \rho g x_i^* \left(2 - \frac{2}{\sqrt{3}} x_i^*\right) \Delta x = \int_{0}^{\sqrt{3}} \rho g x \left(2 - \frac{2}{\sqrt{3}} x\right) dx = \rho g \int_{0}^{\sqrt{3}} \left(2x - \frac{2}{\sqrt{3}} x^2\right) dx
\]
\[
= \rho g \left[ x^2 - \frac{2}{3\sqrt{3}} x^2 \right]_{0}^{\sqrt{3}} = \rho g [(3 - 2) - 0] = \rho g \approx 1000 \cdot 9.8 = 9.8 \times 10^5 \text{ N}
\]

30. (a) The solution is similar to the solution for Example 6. The pressure on a strip is approximately \(\delta d_i = 64.6(3 - y_i^*)\) and the total force is
\[
F = \lim_{n \to \infty} \sum_{i=1}^{n} 64.6(3 - y_i^*)2 \sqrt{9 - (y_i^*)^2} \Delta y = 129.2 \int_{3}^{0} (3 - y) \sqrt{9 - y^2} dy
\]
\[
= 129.2 \cdot 3 \int_{3}^{0} \sqrt{9 - y^2} dy - 129.2 \int_{3}^{0} y \sqrt{9 - y^2} dy
\]
\[
= 387.6 \cdot \frac{1}{2} \pi (3)^2 - 0 \quad \text{[the first integral is the area of a semicircular disk with radius 3 and]}
\]
\[
= 5480 \text{ lb} \quad \text{[the second integral is 0 because the integrand is an odd function]}
\]

(b) If the tank is half full, the surface of the milk is \(y = 0\), so the pressure on a strip is approximately \(\delta d_i = 64.6(0 - y_i^*)\). The upper limit of integration changes from 3 to 0 and the total force is
\[
F = 129.2 \int_{3}^{0} (0 - y) \sqrt{9 - y^2} dy = 129.2 \left[ \frac{1}{3} (9 - y^2)^{3/2} \right]_{3}^{0} = 129.2(9 - 0) = 1162.8 \text{ lb}
\]

Note that this is about 21% of the force for a full tank.

31. By similar triangles, \(\frac{8}{4\sqrt{3}} = \frac{w_i}{x_i^*}\) \(\Rightarrow\) \(w_i = \frac{2x_i^*}{\sqrt{3}}\). The area of the ith rectangular strip is \(\frac{2x_i^*}{\sqrt{3}} \Delta x\) and the pressure on it is \(\rho g \left(4\sqrt{3} - x_i^*\right)\).

\[
F = \int_{0}^{4\sqrt{3}} \rho g \left(4\sqrt{3} - x\right) \frac{2x}{\sqrt{3}} dx = 8 \rho g \int_{0}^{4\sqrt{3}} x dx - \frac{2 \rho g}{\sqrt{3}} \int_{0}^{4\sqrt{3}} x^2 dx
\]
\[
= 4 \rho g \left[ x^2 \right]_{0}^{4\sqrt{3}} - \frac{2 \rho g}{3\sqrt{3}} \left[ \frac{x^3}{3} \right]_{0}^{4\sqrt{3}} = 192 \rho g - \frac{2 \rho g}{3\sqrt{3}} 64 \cdot 3 \sqrt{3} = 192 \rho g - 128 \rho g = 64 \rho g
\]
\[
\approx 64(840)(9.8) \approx 5.27 \times 10^5 \text{ N}
\]
32. The height of the dam is $h = \sqrt{70^2 - 25^2} \cos 30^\circ = 15 \sqrt{19 \left(\frac{\sqrt{2}}{2}\right)}$.

The width of the trapezoid is $w = 50 + 2a$.

By similar triangles, $\frac{25}{h} = \frac{a}{h-x} \Rightarrow a = \frac{25}{h} (h-x)$. Thus,

$w = 50 + 2 \cdot \frac{25}{h} (h-x) = 50 + \frac{50}{h} \cdot h - \frac{50}{h} \cdot x = 50 + 50 - \frac{50x}{h} = 100 - \frac{50x}{h}$.

From the small triangle in the second figure, $\cos 30^\circ = \frac{\Delta x}{z} \Rightarrow z = \Delta x \sec 30^\circ = 2 \Delta x / \sqrt{3}$.

$F = \int_0^h \delta x \left(100 - \frac{50x}{h}\right) \frac{2}{\sqrt{3}} \, dx = \frac{200 \delta}{\sqrt{3}} \int_0^h x \, dx - \frac{100 \delta}{\sqrt{3}} \int_0^h x^2 \, dx = \frac{200h^2}{\sqrt{3}} \cdot \frac{h^3}{3 \sqrt{3}} = \frac{200(62.5)}{3 \sqrt{3}} \cdot \frac{12.825}{4} \approx 7.71 \times 10^6 \text{ lb}$

33. (a) The area of a strip is $10 \Delta x$ and the pressure on it is $\rho g x$.

$F = \int_0^1 \rho g x 10 \, dx = 10 \rho g \left[\frac{1}{2} x^2\right]_0^1 = 10 \rho g \cdot \frac{1}{2} = 5 \rho g$

$= 5(1000)(9.8) = 49000 \text{ N} = 4.9 \times 10^4 \text{ N}$

(b) $F = \int_0^3 \rho g x 10 \, dx = 10 \rho g \left[\frac{1}{2} x^2\right]_0^3 = 10 \rho g \cdot \frac{9}{2} = 45 \rho g = 45(1000)(9.8) = 441000 \text{ N} \approx 4.4 \times 10^6 \text{ N}$.

(c) For the first 1 m, the length of the side is constant at 20 m. For $1 < x \leq 3$, we can use similar triangles to find the length $a$.

$\frac{a}{20} = \frac{3-x}{2} \Rightarrow a = 20 \cdot \frac{3-x}{2} = 10(3-x)$.

$F = \int_0^1 \rho g x 20 \, dx + \int_1^3 \rho g x 10(3-x) \, dx = 20 \rho g \left[\frac{1}{2} x^2\right]_0^1 + 10 \rho g \int_1^3 (3x-x^2) \, dx$

$= 20 \rho g \left(\frac{1}{2}\right) + 10 \rho g \left[\frac{3}{2} x^2 - \frac{1}{3} x^3\right]_1^3 = 10 \rho g + 10 \rho g \left[\left(\frac{27}{2} - 9\right) - \left(\frac{3}{2} - \frac{1}{3}\right)\right]$

$= 10 \rho g + 10 \rho g \left(\frac{10}{3}\right) = \frac{130}{3} \rho g = \frac{130}{3} (1000)(9.8) \text{ N} \approx 424667 \text{ N} \approx 4.2 \times 10^5 \text{ N}$

(d) For any right triangle with hypotenuse on the bottom,

$\sin \theta = \frac{\Delta x}{\text{hypotenuse}} \Rightarrow$

$\text{hypotenuse} = \Delta x \csc \theta = \Delta x \sqrt{\frac{20^2 + 2^2}{2}} = \sqrt{101} \Delta x$.

$F = \int_0^3 \rho g x 10 \sqrt{101} \, dx = 10 \sqrt{101} \rho g \left[\frac{1}{2} x^2\right]_1^3 = 5 \sqrt{101} \rho g (9 - 1)$

$= 40 \sqrt{101}(1000)(9.8) \approx 3939551 \text{ N} \approx 3.9 \times 10^6 \text{ N}$
34. \[ F = \int_{0}^{2} \rho g (10 - x) 2 \sqrt{4 - x^2} \, dx \]
\[ = 20 \rho g \int_{0}^{2} \sqrt{4 - x^2} \, dx - \rho g \int_{0}^{2} \sqrt{4 - x^2} \cdot 2x \, dx \]
\[ = 20 \rho g \frac{1}{2} \pi (2^2) - \rho g \int_{0}^{2} u^{1/2} \, du \quad \left[ u = 4 - x^2, \, du = -2x \, dx \right] \]
\[ = 20 \pi \rho g \left[ u^{1/2} \right]_{0}^{4} = 20 \pi \rho g \left( \frac{16}{3} - \frac{20}{3} \right) \]
\[ = (1000)(9) \left( 20 - 16 \right) \approx 5.63 \times 10^5 \text{ N} \]

35. \[ F' = \int_{0}^{2} \rho g w(x) \, dx \] where \( w(x) \) is the width of the plate at depth \( x \). From the table, we see that \( \Delta x = 0.4 \), so using Simpson’s Rule to estimate \( F' \), we get
\[ F' \approx \frac{4}{3} [7.0w(7.0) + 4(7.4)w(7.4) + 2(7.8)w(7.8) + 4(8.2)w(8.2) + 2(8.6)w(8.6) + 4(9.0)w(9.0) + 9.4w(9.4)] \]
\[ = \frac{28.6}{3} [7.12 + 29.6(1.8) + 15.6(2.9) + 32.8(3.8) + 17.2(3.6) + 36(4.2) + 9.4(4.4)] \]
\[ = \frac{28.6}{3} (486.04) \approx 4148 \text{ lb} \]

36. \( M = m_1 x_1 + m_2 x_2 + m_3 x_3 = 25(-2) + 20(3) + 10(7) = 80, \quad \overline{x} = M/(m_1 + m_2 + m_3) = \frac{80}{101} = \frac{80}{101} \).

37. The mass is \( m = \sum_{i=1}^{3} m_i = 4 + 2 + 4 = 10 \). The moment about the \( x \)-axis is \( M_x = \sum_{i=1}^{3} m_i y_i = 4(-3) + 2(1) + 4(5) = 10 \).

The moment about the \( y \)-axis is \( M_y = \sum_{i=1}^{3} m_i x_i = 4(2) + 2(-3) + 4(3) = 14 \). The center of mass is
\[ (\overline{x}, \overline{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right) = \left( \frac{14}{10}, \frac{10}{10} \right) = (1.4, 1) \).

38. \begin{align*}
M_x &= \sum_{i=1}^{4} m_i x_i = 6(-2) + 5(4) + 1(-7) + 4(-1) = -3, \\
M_y &= \sum_{i=1}^{4} m_i y_i = 6(1) + 5(3) + 1(-3) + 4(6) = 42,
\end{align*}
and \( m = \sum_{i=1}^{4} m_i = 16 \), so \( \overline{x} = \frac{M_y}{m} = \frac{42}{16} = \frac{21}{8} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{-3}{16} \); the center of mass is \((\overline{x}, \overline{y}) = \left( \frac{21}{8}, \frac{-3}{16} \right)\).

39. The region in the figure is “right-heavy” and “bottom-heavy,” so we know that
\begin{align*}
\overline{x} &> 0.5 \quad \text{and} \quad \overline{y} < 1, \quad \text{and we might guess that} \quad \overline{x} = 0.7 \quad \text{and} \quad \overline{y} = 0.7, \\
A &= \int_{0}^{1} 2x \, dx = \left[ x^2 \right]_{0}^{1} = 1 - 0 = 1, \\
\overline{x} &= \frac{1}{A} \int_{0}^{1} x(2x) \, dx = \frac{1}{2} \left[ \frac{2}{3} x^3 \right]_{0}^{1} = \frac{2}{3}, \\
\overline{y} &= \frac{1}{A} \int_{0}^{1} \frac{1}{2} (2x)^2 \, dx = \frac{1}{2} \int_{0}^{1} 2x^2 \, dx = \frac{3}{2}, 
\end{align*}
Thus, the centroid is \((\overline{x}, \overline{y}) = \left( \frac{2}{3}, \frac{3}{2} \right)\).
40. The region in the figure is “right-heavy” and “bottom-heavy,” so we know that $\bar{x} > 2$ and $\bar{y} < 1$, and we might guess that $\bar{x} = 2.3$ and $\bar{y} = 0.8$.

\[
A = \int_0^4 \sqrt{x} \, dx = \left[ \frac{2}{3} x^{3/2} \right]_0^4 = \frac{2 \cdot 2^3}{3} = \frac{16}{3}.
\]

\[
\bar{x} = \frac{1}{A} \int_0^4 x (\sqrt{x}) \, dx = \frac{2}{3} \int_0^4 x^{3/2} \, dx = \frac{2}{16} \left[ \frac{2}{5} x^{5/2} \right]_0^4 = \frac{3}{40} (32 - 0) = \frac{12}{5}.
\]

\[
\bar{y} = \frac{1}{A} \int_0^4 \frac{1}{2} (\sqrt{x})^2 \, dx = \frac{3}{16} \int_0^4 \frac{1}{2} x \, dx = \frac{3}{32} \left[ \frac{1}{2} x^2 \right]_0^4 = \frac{3}{64} (16 - 0) = \frac{3}{4}.
\]

Thus, the centroid is $(\bar{x}, \bar{y}) = (2.4, 0.75)$.

41. The region in the figure is “right-heavy” and “bottom-heavy,” so we know $\bar{x} > 0.5$ and $\bar{y} < 1$, and we might guess that $\bar{x} = 0.6$ and $\bar{y} = 0.9$.

\[
A = \int_0^1 e^x \, dx = [e^x]_0^1 = e - 1.
\]

\[
\bar{x} = \frac{1}{A} \int_0^1 xe^x \, dx = \frac{1}{e - 1} [xe^x - e^x]_0^1 \quad \text{[by parts]}
\]

\[
= \frac{1}{e - 1} [1 - (-1)] = \frac{2}{e - 1}.
\]

\[
\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2} (e^x)^2 \, dx = \frac{1}{2} \int_0^1 e^{2x} \, dx = \frac{1}{2} \int_0^1 [e^x]_0^1 = \frac{1}{2} (e^1 - 1) = \frac{e + 1}{2}.
\]

Thus, the centroid is $(\bar{x}, \bar{y}) = \left( \frac{1}{e - 1}, \frac{e + 1}{2} \right) \approx (0.58, 0.93)$.

42. The region in the figure is “left-heavy” and “bottom-heavy,” so we know $\bar{x} < 1.5$ and $\bar{y} < 0.5$, and we might guess that $\bar{x} = 1.4$ and $\bar{y} = 0.4$.

\[
A = \int_0^2 \frac{1}{x} \, dx = [\ln x]_0^2 = \ln 2.
\]

\[
\bar{x} = \frac{1}{A} \int_0^2 x \cdot \frac{1}{x} \, dx = \frac{1}{A} [\ln x]_0^2 = \frac{1}{\ln 2}.
\]

\[
\bar{y} = \frac{1}{A} \int_0^2 \left( \frac{1}{2} \right)^2 \, dx = \frac{1}{2A} \int_0^2 x^{-2} \, dx = \frac{1}{2A} [-\frac{1}{x}]_1^2
\]

\[
= \frac{1}{2 \ln 2} \left( -\frac{1}{2} + 1 \right) = \frac{1}{2 \ln 2}.
\]

Thus, the centroid is $(\bar{x}, \bar{y}) = \left( \frac{1}{\ln 2}, \frac{1}{2 \ln 2} \right) \approx (1.44, 0.36)$.

43. $A = \int_0^1 (x^{1/2} - x^2) \, dx = \left[ \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1 = \left( \frac{2}{3} - \frac{1}{3} \right) - 0 = \frac{1}{3}$.

\[
\bar{x} = \frac{1}{A} \int_0^1 x (x^{1/2} - x^2) \, dx = \int_0^1 (x^{3/2} - x^3) \, dx
\]

\[
= 3 \left[ \frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right]_0^1 = 3 \left( \frac{2}{5} - \frac{1}{4} \right) = 3 \left( \frac{8}{20} - \frac{5}{20} \right) = \frac{9}{20}.
\]

\[
\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2} \left( (x^{1/2})^2 - (x^2)^2 \right) \, dx = \frac{1}{2} \int_0^1 (x - x^4) \, dx
\]

\[
= \frac{3}{8} \int_0^1 x^3 - \frac{1}{5} x^5 \, dx = \frac{3}{8} \left( \frac{1}{4} - \frac{1}{6} \right) = \frac{3}{8} \left( \frac{3}{20} \right) = \frac{9}{80}.
\]

Thus, the centroid is $(\bar{x}, \bar{y}) = \left( \frac{9}{20}, \frac{9}{80} \right)$. 

7.6-Part 1
44. \( A = \int_{-1}^{2} (x + 2 - x^2) \, dx = \left[ \frac{1}{2} x^2 + 2x - \frac{1}{3} x^3 \right]_{-1} \)
   \[= \left( 2 + 4 - \frac{8}{3} \right) - \left( \frac{1}{2} - 2 + \frac{1}{3} \right) = \frac{9}{3}. \]

\[\bar{x} = \frac{1}{A} \int_{-1}^{2} x(x + 2 - x^2) \, dx = \frac{2}{3} \int_{-1}^{2} (x^2 + 2x - x^3) \, dx \]
\[= \frac{2}{3} \left[ \frac{1}{3} x^3 + x^2 - \frac{1}{3} x^4 \right]_{-1} \]
\[= 3 \left[ \left( \frac{5}{3} + 4 - 1 \right) - (-\frac{1}{3} - 2 + \frac{1}{3}) \right] = \frac{2}{3} \cdot 6 = \frac{4}{3}. \]

\[\bar{y} = \frac{1}{A} \int_{-1}^{2} \frac{1}{2}[(x + 2) - (x^2)] \, dx = \frac{3}{2} \cdot \frac{1}{2} \int_{-1}^{2} (x^2 + 4x + 4 - x^4) \, dx = \frac{1}{9} \left[ \frac{3}{2} x^3 + 2x^2 + 4x - \frac{1}{6} x^5 \right]_{-1} \]
\[= \frac{1}{6} \left[ \left( \frac{5}{3} + 8 + 8 - \frac{32}{9} \right) - \left( -\frac{1}{3} - 2 + 4 + \frac{1}{9} \right) \right] = \frac{1}{6} (18 + \frac{2}{9} - \frac{32}{9}) = \frac{1}{6} \cdot \frac{72}{9} = \frac{8}{9}. \]

Thus, the centroid is \((\bar{x}, \bar{y}) = \left( \frac{4}{3}, \frac{8}{9} \right)\).

45. \( A = \int_{0}^{\pi/4} (\cos x - \sin x) \, dx = \left[ \sin x + \cos x \right]_{0}^{\pi/4} = \sqrt{2} - 1. \)

\[\bar{x} = A^{-1} \int_{0}^{\pi/4} x(\cos x - \sin x) \, dx \]
\[= A^{-1} \left[ x(\sin x + \cos x) + \cos x - \sin x \right]_{0}^{\pi/4} \quad \text{[integration by parts]} \]
\[= A^{-1} (\frac{7}{4} \sqrt{2} - 1) = \frac{\frac{3}{4} \sqrt{2} - 1}{\sqrt{2} - 1}. \]

\[\bar{y} = A^{-1} \int_{0}^{\pi/4} \frac{1}{2}(\cos^2 x - \sin^3 x) \, dx = \frac{1}{2A} \int_{0}^{\pi/4} \cos 2x \, dx = \frac{1}{4A} \left[ \sin 2x \right]_{0}^{\pi/4} = \frac{1}{4A} = \frac{1}{4(\sqrt{2} - 1)}. \]

Thus, the centroid is \((\bar{x}, \bar{y}) = \left( \frac{\pi \sqrt{2} - 4}{4(\sqrt{2} - 1)}, \frac{1}{4(\sqrt{2} - 1)} \right) \approx (0.27, 0.60). \)

46. \( A = \int_{0}^{1} x^2 \, dx + \int_{1}^{2} (2 - x) \, dx = \left[ \frac{1}{3} x^3 \right]_{0}^{1} + \left[ 2x - \frac{1}{3} x^2 \right]_{1} \]
\[= \frac{1}{3} + (4 - 2) - (2 - \frac{1}{3}) = \frac{2}{3}. \]

\[\bar{x} = \frac{1}{A} \left[ \int_{0}^{1} x(x^2) \, dx + \int_{1}^{2} x(2 - x) \, dx \right] = \frac{3}{4} \left[ \int_{0}^{1} x^3 \, dx + \int_{1}^{2} (2x - x^3) \, dx \right] \]
\[= \frac{3}{4} \left\{ \left[ \frac{1}{4} x^4 \right]_{0}^{1} + \left[ x^2 - \frac{1}{3} x^3 \right]_{1} \right\} = \frac{3}{4} \left( \frac{1}{4} + (4 - \frac{8}{3}) - (1 - \frac{1}{3}) \right) \]
\[= \frac{3}{4} \left( \frac{12}{3} \right) = \frac{9}{8}. \]

\[\bar{y} = \frac{1}{A} \left[ \int_{0}^{1} \frac{1}{2} (x^2)^2 \, dx + \int_{1}^{2} \frac{1}{2} (2 - x)^2 \, dx \right] = \frac{3}{2} \left[ \int_{0}^{1} x^6 \, dx + \int_{1}^{2} (x - 2)^2 \, dx \right] = \frac{3}{2} \left\{ \left[ \frac{1}{7} x^7 \right]_{0}^{1} + \left[ \frac{2}{3} (x - 2)^3 \right]_{1} \right\} \]
\[= \frac{3}{2} \left( \frac{1}{7} - 0 + 0 + \frac{1}{9} \right) = \frac{3}{2} \left( \frac{10}{21} \right) = \frac{30}{63}. \]

Thus, the centroid is \((\bar{x}, \bar{y}) = \left( \frac{9}{8}, \frac{30}{63} \right)\).
47. The line has equation \( y = \frac{3}{4} x \). \( A = \frac{1}{2}(4)(3) = 6 \), so \( m = \rho A = 10(6) = 60 \).

\[
M_x = \rho \int_0^4 \left(\frac{3}{4} x\right)^2 \, dx = 10 \int_0^4 \frac{9}{16} x^2 \, dx = \frac{45}{16} \left[ \frac{1}{3} x^3 \right]_0^4 = \frac{45}{16} \left( \frac{54}{3} \right) = 60
\]

\[
M_y = \rho \int_0^4 x \left(\frac{3}{4} x\right) \, dx = \frac{3}{2} \int_0^4 x^2 \, dx = \frac{3}{2} \left[ \frac{1}{3} x^3 \right]_0^4 = \frac{3}{2} \left( \frac{54}{3} \right) = 160
\]

\[
\bar{x} = \frac{M_y}{m} = \frac{160}{60} = \frac{8}{3} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{60}{60} = 1.
\]
Thus, the centroid is \((\bar{x}, \bar{y}) = \left( \frac{8}{3}, 1 \right)\).

48. By symmetry about the line \( y = x \), we expect that \( \bar{x} = \bar{y} \). \( A = \frac{1}{2} \pi r^2 \), so \( m = \rho A = 2A = \frac{1}{2} \pi r^2 \).

\[
M_x = \rho \int_0^r \frac{1}{2} \left( \sqrt{r^2 - x^2} \right)^2 \, dx = 2 \cdot \frac{1}{2} \int_0^r (r^2 - x^2) \, dx = \left[ r^2 x - \frac{1}{2} x^3 \right]_0^r = \frac{3}{2} r^3.
\]

\[
M_y = \rho \int_0^r x \sqrt{r^2 - x^2} \, dx = \int_0^r (r^2 - x^2)^{1/2} 2x \, dx = \int_0^r \sqrt{u} \, du \quad [u = r^2 - x^2] = \left[ \frac{2}{3} u^{3/2} \right]_0^r = \frac{2}{3} r^{3/2}.
\]

\[
\bar{x} = \frac{1}{m} M_y = \frac{2}{\pi r^2} \left( \frac{2}{3} r^{3/2} \right) = \frac{4}{3\pi} r, \quad \bar{y} = \frac{1}{m} M_x = \frac{2}{\pi r^2} \left( \frac{3}{2} r^3 \right) = \frac{4}{3\pi} r.
\]
Thus, the centroid is \((\bar{x}, \bar{y}) = \left( \frac{4}{3\pi} r, \frac{4}{3\pi} r \right)\).
49. Choose $x$- and $y$-axes so that the base (one side of the triangle) lies along the $x$-axis with the other vertex along the positive $y$-axis as shown. From geometry, we know the medians intersect at a point $\frac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from $B$ goes to the midpoint $\left(\frac{1}{2}(a+c), 0\right)$ of side $AC$, so the point of intersection of the medians is $\left(\frac{2}{3} \cdot \frac{1}{2}(a+c), \frac{1}{3}b\right) = \left(\frac{1}{3}(a+c), \frac{1}{3}b\right)$.

This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A = \frac{1}{2}(c-a)b$.

\[
\bar{x} = \frac{1}{A} \left[ \int_{0}^{a} \frac{b}{a} (a-x) \, dx + \int_{0}^{c} \frac{b}{c} (c-x) \, dx \right] = \frac{1}{A} \left[ \frac{b}{a} \left[ \frac{1}{2}a x^2 - \frac{1}{3}x^3 \right]_0^a + \frac{b}{c} \left[ \frac{1}{2}c x^2 - \frac{1}{3}x^3 \right]_0^c \right] = \frac{b}{Aa} \left[ \frac{1}{2}a^2 + \frac{1}{3}a^3 \right] + \frac{b}{Ac} \left[ \frac{1}{2}c^2 + \frac{1}{3}c^3 \right] = \frac{2}{a(c-a)} \cdot \frac{a^2}{6} + \frac{2}{c(c-a)} \cdot \frac{c^2}{6} = \frac{1}{3} \frac{a+c}{a-c} (a^2 + c^2) = \frac{a+c}{3}
\]

and

\[
\bar{y} = \frac{1}{A} \left[ \int_{0}^{a} \frac{1}{2} \left( \frac{b}{a} (a-x) \right)^2 \, dx + \int_{0}^{c} \frac{1}{2} \left( \frac{b}{c} (c-x) \right)^2 \, dx \right] = \frac{1}{A} \left[ \frac{b^2}{2a^2} \int_{0}^{a} (a^2 - 2ax + x^2) \, dx + \frac{b^2}{2c^2} \int_{0}^{c} (c^2 - 2cx + x^2) \, dx \right] = \frac{1}{A} \left[ \frac{b^2}{2a^2} \left[ a^2x - ax^2 + \frac{1}{3}x^3 \right]_0^a + \frac{b^2}{2c^2} \left[ c^2x - cx^2 + \frac{1}{3}x^3 \right]_0^c \right] = \frac{1}{A} \left[ \frac{b^2}{2a^2} (a^2 + c^2) + \frac{b^2}{2c^2} \left( c^2 - a^2 + \frac{1}{3}a^3 \right) \right] = \frac{1}{A} \left[ \frac{b^2}{6} (-a+c) \right] = \frac{2}{(c-a)b} \cdot \frac{(c-a)b^2}{6} = \frac{b}{3}
\]

Thus, the centroid is $(\bar{x}, \bar{y}) = \left(\frac{a+c}{3}, \frac{b}{3}\right)$, as claimed.
Remarks: Actually the computation of \( \overline{y} \) is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is \( \frac{1}{3} \) of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of \( \overline{y} \) in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate \( y \) is \( \ell(y) \), then its area is \( \ell(y) \Delta y \), its mass is \( \rho \ell(y) \Delta y \), and its moment about the \( x \)-axis is \( \Delta M_x = \rho y \ell(y) \Delta y \). Thus,

\[
M_x = \int \rho y \ell(y) \, dy \quad \text{and} \quad \overline{y} = \frac{\int \frac{\rho y \ell(y)}{\rho A} \, dy}{\frac{1}{A} \int y \ell(y) \, dy}
\]

In this problem, \( \ell(y) = \frac{c-a}{b} (b-y) \) by similar triangles, so

\[
\overline{y} = \frac{1}{A} \int_{b}^{c-a} \frac{c-a}{b} y(b-y) \, dy = \frac{2}{b^3} \left[ \frac{1}{2} by^2 - \frac{1}{3} y^3 \right]_{b}^{c-a} = \frac{2}{b^3} \cdot \frac{b^3}{6} = \frac{b}{3}
\]

Notice that only one integral is needed when this method is used.
50. The rectangle to the left of the y-axis has centroid \((-\frac{1}{2}, 1)\) and area 2. The triangle to the right of the y-axis has area 2 and centroid \((\frac{3}{2}, \frac{3}{2})\) [by Exercise 49, the centroid is two-thirds of the way from the vertex \((0, 0)\) to the point \((1, 1)\)].

\[
\bar{x} = \frac{M_y}{m} = \frac{1}{m} \sum_{i=1}^{2} m_i x_i = \frac{1}{2 + 2} \left[2 \left(-\frac{1}{2}\right) + 2 \left(\frac{3}{2}\right)\right] = \frac{1}{4} \left(\frac{1}{2}\right) = \frac{1}{8}.
\]

\[
\bar{y} = \frac{M_x}{m} = \frac{1}{m} \sum_{i=1}^{2} m_i y_i = \frac{1}{2 + 2} \left[2(1) + 2 \left(\frac{3}{2}\right)\right] = \frac{1}{4} \left(\frac{10}{2}\right) = \frac{5}{6}. \text{ Thus, the centroid is } (\bar{x}, \bar{y}) = \left(\frac{1}{8}, \frac{5}{6}\right).
\]

51. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 49, the triangles have centroids \((-1, \frac{3}{2})\) and \((1, \frac{3}{2})\). The centroid of the rectangle (its center) is \((0, -\frac{1}{2})\).

So, using Formulas 9 and 11, we have \(\bar{y} = \frac{M_x}{m} = \frac{1}{m} \sum_{i=1}^{3} m_i y_i = \frac{1}{8} \left[2 \left(\frac{3}{2}\right) + 2 \left(\frac{3}{2}\right) + 4 \left(-\frac{1}{2}\right)\right] = \frac{1}{8} \left(\frac{2}{3}\right) = \frac{1}{12}, \text{ and } \bar{x} = 0,
\]
since the lamina is symmetric about the line \(x = 0\). Thus, the centroid is \((\bar{x}, \bar{y}) = (0, \frac{1}{12})\).

52. A sphere can be generated by rotating a semicircle about its diameter. By Example 8, the center of mass travels a distance

\[
2\pi \bar{y} = 2\pi \left(\frac{4r}{3\pi}\right) = \frac{8r}{3}, \text{ so by the Theorem of Pappus, the volume of the sphere is } V = Ad = \frac{\pi r^2}{2} \cdot \frac{8r}{3} = \frac{4}{3}\pi r^2.
\]

53. A cone of height \(h\) and radius \(r\) can be generated by rotating a right triangle about one of its legs as shown. By Exercise 49, \(\bar{x} = \frac{3}{4}r\), so by the Theorem of Pappus, the volume of the cone is

\[
V = Ad = \left(\frac{1}{2} \cdot \text{base} \cdot \text{height}\right) \cdot (2\pi \bar{x}) = \frac{1}{2}rh \cdot 2\pi \left(\frac{3}{4}r\right) = \frac{3}{4}\pi r^2h.
\]

54. From the symmetry in the figure, \(\bar{y} = 4\). So the distance traveled by the centroid when rotating the triangle about the x-axis is \(d = 2\pi \cdot 4 = 8\pi\). The area of the triangle is \(A = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3\). By the Theorem of Pappus, the volume of the resulting solid is \(Ad = 3(8\pi) = 24\pi\).
55. Suppose the region lies between two curves \( y = f(x) \) and \( y = g(x) \) where \( f(x) \geq g(x) \), as illustrated in Figure 13. Choose points \( x_i \) with \( a = x_0 < x_1 < \cdots < x_n = b \) and choose \( x^*_i \) to be the midpoint of the \( i \)th subinterval; that is, \( x^*_i = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) \). Then the centroid of the \( i \)th approximating rectangle \( R_i \) is its center \( C_i = (\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]) \).

Its area is \( [f(\overline{x}_i) - g(\overline{x}_i)] \Delta x \), so its mass is
\[
\rho[f(\overline{x}_i) - g(\overline{x}_i)] \Delta x \quad \text{Thus,} \quad M_y(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)] \Delta x \cdot \overline{x}_i = \rho \overline{x}_i \left[ f(\overline{x}_i) - g(\overline{x}_i) \right] \Delta x \quad \text{and} \quad M_x(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)] \Delta x \cdot \frac{1}{2} \left[ f(\overline{x}_i) + g(\overline{x}_i) \right] = \rho \cdot \frac{1}{2} \left[ (f(\overline{x}_i))^2 - (g(\overline{x}_i))^2 \right] \Delta x.
\]

Summing over \( i \) and taking the limit as \( n \to \infty \), we get
\[
M_y = \lim_{n \to \infty} \sum_i \rho \overline{x}_i \left[ f(\overline{x}_i) - g(\overline{x}_i) \right] \Delta x = \rho \int_a^b x[f(x) - g(x)] \, dx \quad \text{and} \quad M_x = \lim_{n \to \infty} \sum_i \rho \frac{1}{2} \left[ (f(\overline{x}_i))^2 - (g(\overline{x}_i))^2 \right] \Delta x = \rho \int_a^b \frac{1}{2} \left[ f(x)^2 - g(x)^2 \right] \, dx.
\]

Thus, \( \overline{x} = \frac{M_y}{m} = \frac{M_y}{\rho A} = \frac{1}{A} \int_a^b x[f(x) - g(x)] \, dx \) and \( \overline{y} = \frac{M_x}{m} = \frac{M_x}{\rho A} = \frac{1}{A} \int_a^b \frac{1}{2} \left[ f(x)^2 - g(x)^2 \right] \, dx \).

56. (a) Let \( 0 \leq x \leq 1 \). If \( n < m \), then \( x^n > x^m \); that is, raising \( x \) to a larger power produces a smaller number.

(b) Using Formulas 13 and the fact that the area of \( \mathcal{R} \) is
\[
A = \int_0^1 (x^n - x^m) \, dx = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)},
\]
we get
\[
\overline{x} = \frac{(n+1)(m+1)}{m-n} \int_0^1 x[x^n - x^m] \, dx = \frac{(n+1)(m+1)}{m-n} \int_0^1 (x^{n+1} - x^{m+1}) \, dx = \frac{(n+1)(m+1)}{m-n} \left[ \frac{1}{n+2} - \frac{1}{m+2} \right] = \frac{(n+1)(m+1)}{(n+2)(m+2)}
\]
and
\[
\overline{y} = \frac{(n+1)(m+1)}{m-n} \int_0^1 \frac{1}{2} \left[ (x^n)^2 - (x^m)^2 \right] \, dx = \frac{(n+1)(m+1)}{2(m-n)} \int_0^1 (x^{2n} - x^{2m}) \, dx = \frac{(n+1)(m+1)}{2(m-n)} \left[ \frac{1}{2n+1} - \frac{1}{2m+1} \right] = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}
\]
(c) If we take \( n = 3 \) and \( m = 4 \), then
\[
\left( \overline{x}, \overline{y} \right) = \left( \frac{4 \cdot 5 \cdot 3}{5 \cdot 6 \cdot 7 \cdot 9}, \frac{20 \cdot 26}{3 \cdot 63} \right) = \left( \frac{20}{3}, \frac{20}{63} \right)
\]
which lies outside \( \mathcal{R} \) since \( \left( \frac{2}{3} \right)^2 = \frac{4}{9} < \frac{20}{63} \). This is the simplest of many possibilities.
1. \( \frac{dy}{dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \ln |y| = \ln |x| + C \Rightarrow |y| = e^{\ln |x| + C} = e^{\ln |x|}e^C = e^C |x| \Rightarrow y = Kx, \text{ where } K = \pm e^C \text{ is a constant. (In our derivation, } K \text{ was nonzero, but we can restore the excluded case } y = 0 \text{ by allowing } K \text{ to be zero.)} \\

2. \( \frac{dy}{dx} = \frac{\sqrt{x}}{e^y} \Rightarrow e^y \, dy = \sqrt{x} \, dx \Rightarrow \int e^y \, dy = \int x^{1/2} \, dx \Rightarrow e^y = \frac{2}{3} x^{3/2} + C \Rightarrow y = \ln \left( \frac{2}{3} x^{3/2} + C \right) \)

3. \( xy^2y' = x + 1 \Rightarrow y^2 \frac{dy}{dx} = \frac{x + 1}{x} \Rightarrow y^2 \, dy = \left( 1 + \frac{1}{x} \right) \, dx \Rightarrow \int y^2 \, dy = \int \left( 1 + \frac{1}{x} \right) \, dx \Rightarrow \frac{1}{3} y^3 = x + \ln |x| + C \Rightarrow y^3 = 3x + 3 \ln |x| + 3C \Rightarrow y = \sqrt[3]{3x + 3 \ln |x| + K}, \text{ where } K = 3C. \)

4. \( y' = y^2 \sin x \Rightarrow \frac{dy}{dx} = y^2 \sin x \Rightarrow \frac{dy}{y^2} = \sin x \, dx \Rightarrow \int \frac{dy}{y^2} = \int \sin x \, dx \Rightarrow \frac{1}{y} = -\cos x + C \Rightarrow \frac{1}{y} = \cos x - C \Rightarrow y = \frac{1}{\cos x + K}, \text{ where } K = -C. \text{ } y = 0 \text{ is also a solution.} \)

5. \( (y + \sin y) y' = x + x^3 \Rightarrow (y + \sin y) \frac{dy}{dx} = x + x^3 \Rightarrow \int (y + \sin y) \, dy = \int (x + x^3) \, dx \Rightarrow \frac{1}{3} y^3 - \cos y = \frac{1}{2} x^2 + \frac{1}{4} x^4 + C. \text{ We cannot solve explicitly for } y. \)

6. \( \frac{dy}{d\theta} = \frac{e^y \sin^2 \theta}{y \sec \theta} \Rightarrow \frac{y}{e^y} \, dy = \frac{\sin^2 \theta}{\sec \theta} \, d\theta \Rightarrow \int ye^{-y} \, dy = \int \sin^2 \theta \cos \theta \, d\theta. \text{ Integrating the left side by parts with } u = y, \text{ } dv = e^{-y} \, dy \text{ and the right side by the substitution } u = \sin \theta, \text{ we obtain } -ye^{-y} - e^{-y} = \frac{1}{2} \sin^2 \theta + C. \text{ We cannot solve explicitly for } y. \)

7. \( \frac{dp}{dt} = t^2 p - p + t^2 - 1 = p(t^2 - 1) + 1(t^2 - 1) = (p + 1)(t^2 - 1) \Rightarrow \frac{1}{p + 1} \, dp = (t^2 - 1) \, dt \Rightarrow \int \frac{1}{p + 1} \, dp = \int (t^2 - 1) \, dt \Rightarrow \ln |p + 1| = \frac{1}{2} t^3 - t + C \Rightarrow |p + 1| = e^{t^3/2 - t + C} \Rightarrow p + 1 = \pm e^C e^{t^3/2 - t} \Rightarrow p = K e^{t^3/2 - t} - 1, \text{ where } K = \pm e^C. \text{ Since } p = -1 \text{ is also a solution, } K \text{ can equal 0, and hence, } K \text{ can be any real number.} \)

8. \( \frac{dx}{dt} + e^t + x = 0 \Rightarrow \frac{dx}{dt} = -e^t e^x \Rightarrow \int e^{-x} \, dx = -\int e^t \, dt \Rightarrow -e^{-x} = -e^t + C \Rightarrow e^{-x} = e^t - C \Rightarrow \frac{1}{e^x} = e^t - C \Rightarrow e^x = \frac{1}{e^t - C} \Rightarrow z = \ln \left( \frac{1}{e^t - C} \right) \Rightarrow z = -\ln (e^t - C) \)

9. \( \frac{dy}{dx} = \frac{x}{y} \Rightarrow y \, dy = x \, dx \Rightarrow \int y \, dy = \int x \, dx \Rightarrow \frac{1}{2} y^2 = \frac{1}{2} x^2 + C. \text{ } y(0) = -3 \Rightarrow \frac{1}{2} (-3)^2 = \frac{1}{2} (0)^2 + C \Rightarrow C = \frac{9}{2}, \text{ so } \frac{1}{2} y^2 = \frac{1}{2} x^2 + \frac{9}{2} \Rightarrow y^2 = x^2 + 9 \Rightarrow y = -\sqrt{x^2 + 9} \text{ since } y(0) = -3 < 0. \)
10. \( \frac{dy}{dx} = \frac{y \cos x}{1 + y^2} \), \( y(0) = 1 \). 

\[ dy = y \cos x \, dx \quad \Rightarrow \quad \frac{1 + y^2}{y} \, dy = \cos x \, dx \quad \Rightarrow \quad \int \left( \frac{1}{y} + y \right) \, dy = \int \cos x \, dx \quad \Rightarrow \]

\[ \ln |y| + \frac{1}{2} y^2 = \sin x + C. \quad y(0) = 1 \quad \Rightarrow \quad \ln 1 + \frac{1}{2} y^2 = \sin 0 + C \quad \Rightarrow \quad C = \frac{1}{2}, \text{ so } \ln |y| + \frac{1}{2} y^2 = \sin x + \frac{1}{2}. \]

We cannot solve explicitly for \( y \).

11. \( \frac{du}{dt} = \frac{2t + \sec^2 t}{2u} \), \( u(0) = -5 \). 

\[ \int 2u \, du = \int (2t + \sec^2 t) \, dt \quad \Rightarrow \quad u^2 = t^2 + \tan t + C, \]

where \( |u(0)|^2 = 0^2 + \tan 0 + C \quad \Rightarrow \quad C = (-5)^2 = 25. \) Therefore, \( u^2 = t^2 + \tan t + 25 \), so \( u = \pm \sqrt{t^2 + \tan t + 25} \).

Since \( u(0) = -5 \), we must have \( u = -\sqrt{t^2 + \tan t + 25} \).

12. \( \frac{dP}{dt} = \sqrt{Pt} \quad \Rightarrow \quad \frac{dP}{\sqrt{P}} = \sqrt{t} \, dt \quad \Rightarrow \quad \int P^{-1/2} \, dP = \int t^{1/2} \, dt \quad \Rightarrow \quad 2P^{1/2} = \frac{2}{3} t^{3/2} + C. \)

\[ P(1) = 2 \quad \Rightarrow \quad 2\sqrt{2} = \frac{2}{3} + C \quad \Rightarrow \quad C = 2\sqrt{2} - \frac{2}{3}, \text{ so } 2P^{1/2} = \frac{2}{3} t^{3/2} + 2\sqrt{2} - \frac{2}{3} \quad \Rightarrow \quad \sqrt{P} = \frac{1}{3} t^{3/2} + \sqrt{2} - \frac{1}{3} \quad \Rightarrow \]

\[ P = \left( \frac{1}{3} t^{3/2} + \sqrt{2} - \frac{1}{3} \right)^2. \]

13. \( y' \tan x = a + y, \ 0 < x < \pi/2 \) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a + y}{\tan x} \quad \Rightarrow \quad \frac{dy}{a + y} = \cot x \, dx \quad [a + y \neq 0] \quad \Rightarrow \]

\[ \int \frac{dy}{a + y} = \int \frac{\cos x}{\sin x} \, dx \quad \Rightarrow \quad \ln |a + y| = \ln |\sin x| + C \quad \Rightarrow \quad |a + y| = e^{\ln |\sin x| + C} = e^{\ln |\sin x|} \cdot e^C = e^C |\sin x| \quad \Rightarrow \]

\[ a + y = K \sin x, \text{ where } K = \pm e^C. \] (In our derivation, \( K \) was nonzero, but we can restore the excluded case \( y = -a \) by allowing \( K \) to be zero.) 

\[ y(\pi/3) = a \quad \Rightarrow \quad a + a = K \sin \left( \frac{\pi}{3} \right) \quad \Rightarrow \quad 2a = K \frac{\sqrt{3}}{2} \quad \Rightarrow \quad K = \frac{4a}{\sqrt{3}}. \]

Thus, \( a + y = \frac{4a}{\sqrt{3}} \sin x \) and so \( y = \frac{4a}{\sqrt{3}} \sin x - a \).

14. \( \frac{dL}{dt} = kL^2 \ln t \quad \Rightarrow \quad \frac{dL}{L^2} = k \ln t \, dt \quad \Rightarrow \quad \int \frac{dL}{L^2} = \int k \ln t \, dt \quad \Rightarrow \quad -\frac{1}{L} = kt \ln t - kt + C \quad \Rightarrow \quad L = \frac{1}{kt - kt \ln t - C}. \)

\[ L(1) = -1 \quad \Rightarrow \quad -1 = \frac{1}{k - k \ln 1 - C} \quad \Rightarrow \quad C = k - 1 \quad \Rightarrow \quad C = k + 1. \] Thus, \( L = \frac{1}{kt - kt \ln t - k - 1}. \)

15. If the slope at the point \( (x, y) \) is \( xy \), then we have \( \frac{dy}{dx} = xy \quad \Rightarrow \quad \frac{dy}{y} = x \, dx \quad [y \neq 0] \quad \Rightarrow \quad \int \frac{dy}{y} = \int x \, dx \quad \Rightarrow \]

\[ \ln |y| = \frac{1}{2} x^2 + C. \quad y(0) = 1 \quad \Rightarrow \quad \ln 1 = 0 + C \quad \Rightarrow \quad C = 0. \] Thus, \( |y| = e^{x^2/2} \quad \Rightarrow \quad y = \pm e^{x^2/2}, \text{ so } y = e^{x^2/2} \) since \( y(0) = 1 > 0 \). Note that \( y = 0 \) is not a solution because it doesn't satisfy the initial condition \( y(0) = 1 \).
16. \( f'(x) = f(x)(1 - f(x)) \implies \frac{dy}{dx} = y(1 - y) \implies \frac{dy}{y(1 - y)} = dx \quad [y \neq 0, 1] \implies \int \frac{dy}{y(1 - y)} = \int dx \implies \int \left( \frac{A}{y} + \frac{B}{1-y} \right) dy = \int dx \implies \int \left( \frac{1}{y} + \frac{1}{1-y} \right) dy = \int dx \implies \ln |y| - \ln |1 - y| = x + C \implies \\
\ln \left| \frac{y}{1-y} \right| = x + C \implies \frac{y}{1-y} = e^{x+C} \implies \frac{y}{1-y} = Ke^x, \text{ where } K = \pm e^C \implies \\
y = (1-y)Ke^x = Ke^x - yKe^x \implies y + yKe^x = Ke^x \implies y(1 + Ke^x) = Ke^x \implies y = \frac{Ke^x}{1 + Ke^x}.

f(0) = \frac{1}{2} \implies \frac{1}{2} = \frac{K}{1+K} \implies 1 + K = 2K \implies K = 1, \text{ so } y = \frac{e^x}{1+e^x} \left[ \text{or } \frac{1}{1+e^{-x}} \right].

Note that \( y = 0 \) and \( y = 1 \) are not solutions because they don’t satisfy the initial condition \( f(0) = \frac{1}{2} \).

17. (a) \( y' = 2x \sqrt{1-y^2} \implies \frac{dy}{dx} = 2x \sqrt{1-y^2} \implies \frac{dy}{\sqrt{1-y^2}} = 2x \ dx \implies \int \frac{dy}{\sqrt{1-y^2}} = \int 2x \ dx \implies \\
\sin^{-1} y = x^2 + C \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.

(b) \( y(0) = 0 \implies \sin^{-1} 0 = 0^2 + C \implies C = 0, \)

so \( \sin^{-1} y = x^2 \) and \( y = \sin(x^2) \) for \(-\sqrt{\pi}/2 \leq x \leq \sqrt{\pi}/2.\)

(c) For \( \sqrt{1-y^2} \) to be a real number, we must have \(-1 \leq y \leq 1\); that is, \(-1 \leq y(0) \leq 1\). Thus, the initial-value problem \( y' = 2x \sqrt{1-y^2}, y(0) = 2 \) does not have a solution.
18. \( e^{-y}y' + \cos x = 0 \) \( \iff \int e^{-y} dy = -\int \cos x \, dx \) \( \iff -e^{-y} = -\sin x + C_1 \) \( \iff y = -\ln(\sin x + C) \). The solution is periodic, with period \( 2\pi \). Note that for \( C > 1 \), the domain of the solution is \( \mathbb{R} \), but for \(-1 < C \leq 1 \) it is only defined on the intervals where \( \sin x + C > 0 \), and it is meaningless for \( C \leq -1 \), since then \( \sin x + C \leq 0 \), and the logarithm is undefined.

For \(-1 < C < 1 \), the solution curve consists of concave-up pieces separated by intervals on which the solution is not defined (where \( \sin x + C \leq 0 \)). For \( C = 1 \), the solution curve consists of concave-up pieces separated by vertical asymptotes at the points where \( \sin x + C = 0 \) \( \iff \sin x = -1 \). For \( C > 1 \), the curve is continuous, and as \( C \) increases, the graph moves downward, and the amplitude of the oscillations decreases.

19. \( \frac{dy}{dx} = -\frac{\sin x}{\sin y} \) \( y(0) = \frac{\pi}{2} \). So \( \int \sin y \, dy = \int \sin x \, dx \) \( \iff -\cos y = -\cos x + C \) \( \iff \cos y = \cos x - C \). From the initial condition, we need \( \cos \frac{\pi}{2} = \cos 0 - C \) \( \iff 0 = 1 - C \) \( \iff C = 1 \), so the solution is \( \cos y = \cos x - 1 \). Note that we cannot take \( \cos^{-1} \) of both sides, since that would unnecessarily restrict the solution to the case where \(-1 \leq \cos x - 1 \) \( \iff 0 \leq \cos x \), as \( \cos^{-1} \) is defined only on \([-1, 1]\). Instead we plot the graph using Maple’s plots[implicitplot] or Mathematica’s Plot[Evaluate[...]].
20. \( \frac{dy}{dx} = \frac{x \sqrt{x^2 + 1}}{y e^y} \) \( \Rightarrow \) \( \int y e^y \, dy = \int x \sqrt{x^2 + 1} \, dx \). We use parts on the LHS with \( u = y \), \( dv = e^y \, dy \), and on the RHS we use the substitution \( z = x^2 + 1 \), so \( dz = 2x \, dx \). The equation becomes \( y e^y - \int e^y \, dy = \frac{1}{2} \int \sqrt{z} \, dz \) \( \Rightarrow \) \( e^y (y - 1) = \frac{1}{6} (x^2 + 1)^{3/2} + C \), so we see that the curves are symmetric about the \( y \)-axis. Every point \((x, y)\) in the plane lies on one of the curves, namely the one for which \( C = (y - 1)e^y - \frac{1}{6}(x^2 + 1)^{3/2} \). For example, along the \( y \)-axis, \( C = (y - 1)e^y - \frac{1}{6} \), so the origin lies on the curve with \( C = -\frac{2}{3} \). We use Maple's plots [implicitplot] command or Plot [Evaluate[\ldots]] in Mathematica to plot the solution curves for various values of \( C \).

![Graphs of solution curves for different values of C.](Image)

It seems that the transitional values of \( C \) are \(-\frac{4}{3}\) and \(-\frac{1}{2}\). For \( C < -\frac{4}{3} \), the graph consists of left and right branches. At \( C = -\frac{4}{3} \), the two branches become connected at the origin, and as \( C \) increases, the graph splits into top and bottom branches.

At \( C = -\frac{1}{2} \), the bottom half disappears. As \( C \) increases further, the graph moves upward, but doesn't change shape much.

21. \( y' = 2 - y \). The slopes at each point are independent of \( x \), so the slopes are the same along each line parallel to the \( x \)-axis.

Thus, III is the direction field for this equation. Note that for \( y = 2 \), \( y' = 0 \).
22. \( y' = x(2 - y) = 0 \) on the lines \( x = 0 \) and \( y = 2 \). Direction field I satisfies these conditions.

23. \( y' = x + y - 1 = 0 \) on the line \( y = -x + 1 \). Direction field IV satisfies this condition. Notice also that on the line \( y = -x \) we have \( y' = -1 \), which is true in IV.

24. \( y' = \sin x \sin y = 0 \) on the lines \( x = 0 \) and \( y = 0 \), and \( y' > 0 \) for \( 0 < x < \pi \), \( 0 < y < \pi \). Direction field II satisfies these conditions.

25. (a) \( y(0) = 1 \)
    (b) \( y(0) = 2 \)
    (c) \( y(0) = -1 \)

26. (a) \( y(0) = -1 \)
    (b) \( y(0) = 0 \)
    (c) \( y(0) = 1 \)

27. | \( x \) | \( y \) | \( y' = \frac{1}{2}y \) |
    |-----|-----|---------------|
    | 0   | 0   | 0             |
    | 0   | 1   | 0.5           |
    | 0   | 2   | 1             |
    | 0   | -3  | -1.5          |
    | 0   | -2  | -1            |

Note that for \( y = 0 \), \( y' = 0 \). The three solution curves sketched go through \((0, 0)\), \((0, 1)\), and \((0, -1)\).
28. \[ x \quad y \quad y' = x^2 - y^2 \\
\pm 1 \quad \pm 3 \quad -8 \\
\pm 3 \quad \pm 1 \quad 8 \\
\pm 1 \quad \pm 0.5 \quad 0.75 \\
\pm 0.5 \quad \pm 1 \quad -0.75 \\
\]
Note that \( y' = 0 \) for \( y = \pm x \). If \( |x| < |y| \), then \( y' < 0 \), that is, the slopes are negative for all points in quadrants I and II above both of the lines \( y = x \) and \( y = -x \), and all points in quadrants III and IV below both of the lines \( y = -x \) and \( y = x \). A similar statement holds for positive slopes.

29. \[ x \quad y \quad y' = y - 2x \\
-2 \quad -2 \quad 2 \\
-2 \quad 2 \quad 6 \\
2 \quad 2 \quad -2 \\
2 \quad -2 \quad -6 \\
\]
Note that \( y' = 0 \) for any point on the line \( y = 2x \). The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through \((1, 0)\)

30. \[ x \quad y \quad y' = 1 - xy \\
\pm 1 \quad \pm 1 \quad 0 \\
\pm 2 \quad \pm 2 \quad -3 \\
\mp 2 \quad \mp 2 \quad 5 \\
\]
Note that \( y' = 0 \) for any point on the hyperbola \( xy = 1 \) (or \( y = 1/x \)). The slopes are negative at points “inside” the branches and positive at points everywhere else. The solution curve in the graph passes through \((0, 0)\).
31. \[
\begin{array}{c|c|c}
  x & y & y' = y + xy \\
  \hline
  0 & \pm 2 & \pm 2 \\
  1 & \pm 2 & \pm 4 \\
  -3 & \pm 2 & \mp 4 \\
\end{array}
\]
Note that \( y' = y(x + 1) = 0 \) for any point on \( y = 0 \) or on \( x = -1 \).
The slopes are positive when the factors \( y \) and \( x + 1 \) have the same sign and negative when they have opposite signs. The solution curve in the graph passes through \((0, 1)\).

32. \[
\begin{array}{c|c|c}
  x & y & y' = x - xy \\
  \hline
  \pm 2 & 0 & \pm 2 \\
  \pm 2 & 3 & \pm 4 \\
  \pm 2 & -1 & \pm 4 \\
\end{array}
\]
Note that \( y' = x(1 - y) = 0 \) for any point on \( x = 0 \) or on \( y = 1 \). The slopes are positive when the factors \( x \) and \( 1 - y \) have the same sign and negative when they have opposite signs. The solution curve in the graph passes through \((1, 0)\).

33. (a) \( \frac{dP}{dt} = k(M - P) \) is always positive, so the level of performance \( P \) is increasing. As \( P \) gets close to \( M \), \( dP/dt \) gets close to 0; that is, the performance levels off.

(b) \( \frac{dP}{dt} = k(M - P) \Leftrightarrow \int \frac{dP}{P - M} = \int (-k) \, dt \Leftrightarrow \ln|P - M| = -kt + C \Leftrightarrow |P - M| = e^{-kt+C} \Leftrightarrow P - M = Ae^{-kt} \) [\( A = \pm e^C \)] \Leftrightarrow \( P = M + Ae^{-kt} \). If we assume that performance is at level 0 when \( t = 0 \), then
\[
P(0) = 0 \Leftrightarrow 0 = M + A \Leftrightarrow A = -M \Leftrightarrow P(t) = M - Me^{-kt}. \quad \lim_{t \to \infty} P(t) = M - M \cdot 0 = M.
\]
34. If \( S = \frac{dT}{dr} \), then \( \frac{dS}{dr} = \frac{d^2T}{dr^2} \). The differential equation \( \frac{d^2T}{dr^2} + \frac{2}{r} \frac{dT}{dr} = 0 \) can be written as \( \frac{dS}{dr} + \frac{2}{r} S = 0 \). Thus,

\[
\frac{dS}{dr} = -\frac{2S}{r} \quad \Rightarrow \quad \frac{dS}{S} = -\frac{2}{r} \, dr \quad \Rightarrow \quad \int \frac{1}{S} \, dS = \int -\frac{2}{r} \, dr \quad \Rightarrow \quad \ln|S| = -2 \ln|r| + C.
\]

Assuming \( S = \frac{dT}{dr} > 0 \) and \( r > 0 \), we have \( S = e^{-2 \ln r + C} = e^{\ln r^{-2}} e^C = r^{-2} k \) \( [k = e^C] \) \( \Rightarrow \quad S = \frac{1}{r^2} k \quad \Rightarrow \quad \frac{dT}{dr} = \frac{1}{r^2} k \quad \Rightarrow \quad dT = \frac{1}{r^2} k \, dr \quad \Rightarrow \quad T(r) = \frac{k}{r} + A \).

\( T(1) = 15 \) \( \Rightarrow \quad 15 = -k + A \) (1) and \( T(2) = 25 \) \( \Rightarrow \quad 25 = -\frac{1}{2} k + A \) (2).

Now solve for \( k \) and \( A \):

\( -2(2) + 1 \) \( \Rightarrow \quad -35 = -A \), so \( A = 35 \) and \( k = 20 \), and \( T(r) = -20/r + 35 \).

35. (a) \( \frac{dC}{dt} = r - kC \quad \Rightarrow \quad \frac{dC}{kC - r} = -\frac{(kC - r)}{r} \quad \Rightarrow \quad \int \frac{dC}{kC - r} = \int -\frac{dt}{t} \quad \Rightarrow \quad \ln|kC - r| = -kt + M_2 \quad \Rightarrow \quad |kC - r| = e^{-kt + M_2} \quad \Rightarrow \quad kC - r = M_2 e^{-kt} \quad \Rightarrow \quad kC = M_2 e^{-kt} + r \quad \Rightarrow \quad C(t) = M_2 e^{-kt} + r/k \quad \Rightarrow \quad C(0) = C_0 \quad \Rightarrow \quad C_0 = M_2 + r/k \quad \Rightarrow \quad M_2 = C_0 - r/k \quad \Rightarrow \quad C(t) = (C_0 - r/k) e^{-kt} + r/k \).

(b) If \( C_0 < r/k \), then \( C_0 - r/k < 0 \) and the formula for \( C(t) \) shows that \( C(t) \) increases and \( \lim_{t \to \infty} C(t) = r/k \).

As \( t \) increases, the formula for \( C(t) \) shows how the role of \( C_0 \) steadily diminishes as that of \( r/k \) increases.

36. (a) Use 1 billion dollars as the \( x \)-unit and 1 day as the \( t \)-unit. Initially, there is $10 billion of old currency in circulation, so all of the $50 million returned to the banks is old. At time \( t \), the amount of new currency is \( x(t) \) billion dollars, so 10 - \( x(t) \) billion dollars of currency is old. The fraction of circulating money that is old is \( 10 - x(t) \)/10, and the amount of old currency being returned to the banks each day is \( \frac{10 - x(t)}{10} \) 0.05 billion dollars. This amount of new currency per day is introduced into circulation, so \( \frac{dx}{10} = \frac{10 - x}{10} \cdot 0.05 = 0.005(10 - x) \) billion dollars per day.

\[
\frac{dx}{10-x} = 0.005 \quad \Rightarrow \quad \frac{-dx}{10-x} = -0.005 \quad \Rightarrow \quad \ln(10-x) = -0.005 t + e \quad \Rightarrow \quad 10 - x = Ce^{-0.005t},
\]

where \( C = e^e \quad \Rightarrow \quad x(t) = 10 - Ce^{-0.005t} \). From \( x(0) = 0 \), we get \( C = 10 \), so \( x(t) = 10(1 - e^{-0.005t}) \).

(c) The new bills make up 90% of the circulating currency when \( x(t) = 0.9 \cdot 10 = 9 \) billion dollars.

\[
9 = 10(1 - e^{-0.005t}) \quad \Rightarrow \quad 0.9 = 1 - e^{-0.005t} \quad \Rightarrow \quad e^{-0.005t} = 0.1 \quad \Rightarrow \quad -0.005t = -\ln(0.1) \quad \Rightarrow \quad t = 200 \ln 10 \approx 460.517 \text{ days} \approx 1.26 \text{ years}.
\]
37. The differential equation is a logistic equation with \( k = 0.00008 \), carrying capacity \( M = 1000 \), and initial population \( y_0 = P(0) = 100 \). So Equation 8 gives the population at time \( t \) as

\[
P(t) = \frac{100 \cdot 1000}{100 + (1000 - 100)e^{-0.08t}} = \frac{100,000}{100 + 900e^{-0.08t}} = \frac{1000}{1 + 9e^{-0.08t}}
\]

So the population sizes when \( t = 40 \) and \( 80 \) are

\[
P(40) = \frac{1000}{1 + 9e^{-0.08 \cdot 40}} \approx 731.6 \quad P(80) = \frac{1000}{1 + 9e^{-0.08 \cdot 80}} \approx 985.3
\]

The population reaches 900 when \( \frac{1000}{1 + 9e^{-0.08t}} = 900 \). Solving this equation for \( t \), we get \( 1 + 9e^{-0.08t} = \frac{10}{9} \Rightarrow e^{-0.08t} = \frac{1}{81} \Rightarrow -0.08t = \ln \frac{1}{81} = -\ln 81 \Rightarrow t = \frac{\ln 81}{0.08} \approx 54.9 \). So the population reaches 900 when \( t \) is approximately 55.

38. (a) \( \frac{dy}{dt} = ky(M - y) \Rightarrow y(t) = \frac{y_0 M}{y_0 + (M - y_0)e^{-kM t}} \) by (8). With \( M = 8 \times 10^7 \), \( k = 8.875 \times 10^{-9} \), and \( y_0 = 2 \times 10^7 \), we get the model \( y(t) = \frac{(2 \times 10^7)(8 \times 10^7)}{2 \times 10^7 + (6 \times 10^7)e^{-0.71t}} = \frac{8 \times 10^7}{1 + 3e^{-0.71t}} \), so

\[
y(1) = \frac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7 \text{ kg.}
\]

(b) \( y(t) = 4 \times 10^7 \Rightarrow \frac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \Rightarrow 2 = 1 + 3e^{-0.71t} \Rightarrow e^{-0.71t} = \frac{1}{3} \Rightarrow -0.71t = \ln \frac{1}{3} \Rightarrow t = \frac{\ln 3}{0.71} \approx 1.55 \text{ years} \)
39. (a) Our assumption is that \( \frac{dy}{dt} = ky(1 - y) \), where \( y \) is the fraction of the population that has heard the rumor.

(b) The equation in part (a) is the logistic differential equation (7) with \( M = 1 \), so the solution is given by (8):

\[
y = \frac{y_0}{y_0 + (1 - y_0)e^{-kt}}.
\]

(c) Let \( t \) be the number of hours since 8 AM. Then \( y_0 = y(0) = \frac{60}{1000} = 0.06 \) and \( y(4) = \frac{1}{2} \), so

\[
\frac{1}{2} = y(4) = \frac{0.08}{0.08 + 0.92e^{-4k}}.
\]

Thus, \( 0.08 + 0.92e^{-4k} = 0.16, e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23} \), and \( e^{-k} = \left( \frac{2}{23} \right)^{1/4} \).

So

\[
y = \frac{0.08}{0.08 + 0.92(2/23)^{1/4}} = \frac{2}{2 + 23(2/23)^{1/4}}.
\]

Solving this equation for \( t \), we get

\[
2y + 23y \left( \frac{2}{23} \right)^{t/4} = 2 \Rightarrow \left( \frac{2}{23} \right)^{t/4} = \frac{2 - 2y}{23y} \Rightarrow \left( \frac{2}{23} \right)^{t/4} = \frac{2}{23} \cdot \frac{1 - y}{y} \Rightarrow \left( \frac{2}{23} \right)^{t/4 - 1} = \frac{1 - y}{y}.
\]

It follows that \( \frac{t}{4} - 1 = \frac{\ln(1 - y)/y}{\ln \frac{2}{23}} \), so \( t = 4 \left[ 1 + \frac{\ln(1 - y)/y}{\ln \frac{2}{23}} \right] \).

When \( y = 0.9, \frac{1 - y}{y} = \frac{1}{9} \), so \( t = 4 \left( 1 - \frac{\ln 9}{\ln \frac{2}{23}} \right) \approx 7.6 \) h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

40. (a) \( P(0) = P_0 = 400, P(1) = 1200 \) and \( M = 10,000 \). From the solution to the logistic differential equation

\[
P(t) = \frac{P_0M}{P_0 + (M - P_0)e^{-kt}}, \text{ we get } P = \frac{400 \cdot 10000}{400 + (9600) e^{-kt}} = \frac{10000}{1 + 24e^{-kt}}. \quad P(1) = 1200 \Rightarrow
\]

\[
1 + 24e^{-k} = \frac{1000}{12} \Rightarrow e^k = \frac{200}{23} \Rightarrow k = \ln \frac{200}{23}. \quad \text{So } P = \frac{10000}{1 + 24e^{-k}\ln(200/23)} = \frac{10000}{1 + 24 \cdot (11/36)^t}.
\]

(b) \( 5000 = \frac{10000}{1 + 24(11/36)^t} \Rightarrow 24(11/36)^t = 1 \Rightarrow t \ln \frac{11}{36} = \ln \frac{1}{24} \Rightarrow t \approx 2.68 \) years.

41. (a) \( \frac{dy}{dt} = ky(M - y) \Rightarrow \)

\[
\frac{d^2y}{dt^2} = k \left( -\frac{dy}{dt} \right) + k(M - y) \frac{dy}{dt} = k^2 \frac{dy}{dt} (M - 2y) = k^2(y(M - y))(M - 2y).
\]

(b) \( y \) grows fastest when \( y' \) has a maximum, that is, when \( y'' = 0 \). From part (a), \( y'' = 0 \iff y = 0, y = M, \) or \( y = M/2 \).

Since \( 0 < y < M \), we see that \( y'' = 0 \iff y = M/2 \).
42. First we keep $k$ constant (at 0.1, say) and change $y_0$ in the function $y = \frac{10y_0}{y_0 + (10 - y_0)e^{-t}}$. (Notice that $y_0$ is the $y$-intercept.) If $y_0 = 0$, the function is 0 everywhere. For $0 < y_0 < 5$, the curve has an inflection point, which moves to the right as $y_0$ decreases. If $5 < y_0 < 10$, the graph is concave down everywhere. (We are considering only $t \geq 0$.) If $y_0 = 10$, the function is the constant function $y = 10$, and if $y_0 > 10$, the function decreases. For all $y_0 \neq 0$, $\lim_{t \to \infty} y = 10$.

Now we instead keep $y_0$ constant (at $y_0 = 1$) and change $k$ in the function $y = \frac{10}{1 + 9e^{-10kt}}$. It seems that as $k$ increases, the graph approaches the line $y = 10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable $x$-scales, the graphs all look the same.)

43. (a) Let $y(t)$ be the amount of salt (in kg) after $t$ minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time $t$ (in minutes) is $y(t)/1000$ kg/L and

$$\frac{dy}{dt} = -\left[\frac{y(t)}{1000} \text{ kg/L}\right] \left(10 \text{ L/min}\right) = -\frac{y(t)}{100} \text{ kg/min}.$$

$$\int \frac{dy}{y} = -\frac{1}{100} \int dt \quad \Rightarrow \quad \ln y = -\frac{t}{100} + C, \quad \text{and} \quad y(0) = 15 \quad \Rightarrow \quad \ln 15 = C,$$

so $\ln y = \ln 15 - \frac{t}{100}$.

It follows that $\ln \left(\frac{y}{15}\right) = -\frac{t}{100}$ and $\frac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.
44. Let \( y(t) \) be the amount of carbon dioxide in the room after \( t \) minutes. Then \( y(0) = 0.0015(180) = 0.27 \text{ m}^3 \). The amount of air in the room is \( 180 \text{ m}^3 \) at all times, so the percentage at time \( t \) (in minutes) is \( y(t)/180 \times 100 \), and the change in the amount of carbon dioxide with respect to time is

\[
\frac{dy}{dt} = (0.0005) \left( \frac{2 \text{ m}^3}{\text{min}} \right) - \frac{y(t)}{180} \left( \frac{2 \text{ m}^3}{\text{min}} \right) = 0.001 - \frac{y}{90} = \frac{9 - 100y}{9000} \frac{\text{m}^3}{\text{min}}
\]

Hence, \( \int \frac{dy}{9 - 100y} = \int \frac{dt}{9000} \) and \( -\frac{1}{100} \ln |9 - 100y| = \frac{1}{9000} t + C \). Because \( y(0) = 0.27 \), we have

\[
-\frac{1}{100} \ln 18 = C, \quad \text{so} \quad -\frac{1}{100} \ln |9 - 100y| = -\frac{1}{100} \frac{1}{9000} t + \ln 18 = -\frac{t}{90} + \ln 18 \Rightarrow \ln |9 - 100y| = \ln(18e^{-t/90}), \quad \text{and} \quad |9 - 100y| = 18e^{-t/90}. \]

Since \( y \) is continuous, \( y(0) = 0.27 \), and the right-hand side is never zero, we deduce that \( 9 - 100y \) is always negative. Thus, \( |9 - 100y| = 100y - 9 \) and we have \( 100y - 9 = 18e^{-t/90} \Rightarrow 100y = 9 + 18e^{-t/90} \Rightarrow y = 0.09 + 0.18e^{-t/90}. \) The percentage of carbon dioxide in the room is

\[
p(t) = \frac{y}{180} \times 100 = \frac{0.09 + 0.18e^{-t/90}}{180} \times 100 = (0.0005 + 0.001e^{-t/90}) \times 100 = 0.05 + 0.1e^{-t/90}_n\]

In the long run, we have \( \lim_{t \to \infty} p(t) = 0.05 + 0.1(0) = 0.05 \); that is, the amount of carbon dioxide approaches 0.05% as time goes on.

45. Let \( y(t) \) be the amount of alcohol in the vat after \( t \) minutes. Then \( y(0) = 0.04(2000) = 80 \text{ L} \). The amount of beer in the vat is \( 2000 \text{ L} \) at all times, so the percentage at time \( t \) (in minutes) is \( y(t)/2000 \times 100 \), and the change in the amount of alcohol with respect to time \( t \) is

\[
\frac{dy}{dt} = \text{rate in} - \text{rate out} = 0.06 \left( \frac{20 \text{ L}}{\text{min}} \right) - \frac{y(t)}{2000} \left( \frac{20 \text{ L}}{\text{min}} \right) = 1.2 - \frac{y}{100} = \frac{120 - y}{100} \frac{\text{L}}{\text{min}}
\]

Hence, \( \int \frac{dy}{120 - y} = \int \frac{dt}{100} \) and \( -\ln |120 - y| = \frac{1}{100} t + C \). Because \( y(0) = 80 \), we have \( -\ln 40 = C \), so

\[
-\ln |120 - y| = -\frac{1}{100} t - \ln 40 \Rightarrow \ln |120 - y| = -t/100 + \ln 40 \Rightarrow \ln |120 - y| = \ln e^{-t/100} + \ln 40 \Rightarrow \ln |120 - y| = \ln (40e^{-t/100}) \Rightarrow |120 - y| = 40e^{-t/100}. \]

Since \( y \) is continuous, \( y(0) = 80 \), and the right-hand side is never zero, we deduce that \( 120 - y \) is always positive. Thus, \( 120 - y = 40e^{-t/100} \Rightarrow y = 120 - 40e^{-t/100}. \) The percentage of alcohol is \( p(t) = y(t)/2000 \times 100 = y(t)/20 = 6 - 2e^{-t/100}. \) The percentage of alcohol after one hour is \( p(60) = 6 - 2e^{-60/100} \approx 4.9. \)
46. (a) If \( y(t) \) is the amount of salt (in kg) after \( t \) minutes, then \( y(0) = 0 \) and the total amount of liquid in the tank remains constant at 1000 L.

\[
\frac{dy}{dt} = \left(0.05 \text{ kg/L} \right) \left(5 \text{ L/min} \right) + \left(0.04 \text{ kg/L} \right) \left(10 \text{ L/min} \right) - \left(\frac{y(t)}{1000} \text{ kg/L} \right) \left(15 \text{ L/min} \right)
\]

\[= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200} \text{ kg/min} \]

Hence, \( \int \frac{dy}{130 - 3y} = \int \frac{dt}{200} \) and \(-\frac{1}{3} \ln |130 - 3y| = \frac{1}{200} t + C \). Because \( y(0) = 0 \), we have \(-\frac{1}{3} \ln 130 = C \), so

\[-\frac{1}{3} \ln |130 - 3y| = \frac{1}{200} t - \frac{1}{3} \ln 130 \Rightarrow \ln |130 - 3y| = -\frac{2}{200} t + \ln 130 = \ln(130e^{-3t/200}) \], and

\[|130 - 3y| = 130e^{-3t/200}. \] Since \( y \) is continuous, \( y(0) = 0 \), and the right-hand side is never zero, we deduce that \( 130 - 3y \) is always positive. Thus, \( 130 - 3y = 130e^{-3t/200} \) and \( y = \frac{130}{3} (1 - e^{-3t/200}) \) kg.

(b) After one hour, \( y = \frac{130}{3} (1 - e^{-3\cdot60/200}) = \frac{130}{3} (1 - e^{-0.9}) \approx 25.7 \) kg.

Note: As \( t \to \infty \), \( y(t) \to \frac{130}{3} = 43\frac{1}{3} \) kg.

47. Assume that the raindrop begins at rest, so that \( v(0) = 0 \). \( \frac{dm}{dt} = km \) and \( (mv)' = gm \Rightarrow mv' + vm' = gm \Rightarrow \]

\[mv' + v(km) = gm \Rightarrow v' + vk = g \Rightarrow \frac{dv}{dt} = g - kv = \int \frac{dv}{g - kv} = \int dt \Rightarrow \]

\[-\frac{1}{k} \ln |g - kv| = t + C \Rightarrow \ln |g - kv| = -kt - kC \Rightarrow g - kv = Ae^{-kt}. \] \( v(0) = 0 \) \( \Rightarrow A = g \).

So \( kv = g - ge^{-kt} \Rightarrow v = (g/k)(1 - e^{-kt}). \) Since \( k > 0 \), as \( t \to \infty, e^{-kt} \to 0 \) and therefore, \( \lim_{t \to \infty} v(t) = g/k \).
48. (a) \( \frac{dv}{dt} = -kv \Rightarrow \frac{dv}{v} = -\frac{k}{m} \, dt \Rightarrow \ln |v| = -\frac{k}{m} t + C. \) Since \( v(0) = v_0, \ln |v_0| = C. \) Therefore,

\[ \ln \left| \frac{v}{v_0} \right| = -\frac{k}{m} t \Rightarrow \left| \frac{v}{v_0} \right| = e^{-\frac{k}{m} t} \Rightarrow \quad v(t) = \pm v_0 e^{-\frac{k}{m} t}. \] The sign is + when \( t = 0, \) and we assume \( v \) is continuous, so that the sign is + for all \( t. \) Thus, \( v(t) = v_0 e^{-\frac{k}{m} t}. \) \( ds/dt = v_0 e^{-\frac{k}{m} t} \Rightarrow \)

\[ s(t) = -\frac{mv_0}{k} e^{-\frac{k}{m} t} + C'. \]

From \( s(0) = s_0, \) we get \( s_0 = -\frac{mv_0}{k} + C', \) so \( C' = s_0 + \frac{mv_0}{k} \) and \( s(t) = s_0 + \frac{mv_0}{k} \left( 1 - e^{-\frac{k}{m} t} \right). \)

The distance traveled from time 0 to time \( t \) is \( s(t) - s_0, \) so the total distance traveled is \( \lim_{t \to \infty} [s(t) - s_0] = \frac{mv_0}{k}. \)

Note: In finding the limit, we use the fact that \( k > 0 \) to conclude that \( \lim_{t \to \infty} e^{-\frac{k}{m} t} = 0. \)

(b) \( m \frac{dv}{dt} = -kv^2 \Rightarrow \frac{dv}{v^2} = -\frac{k}{m} \, dt \Rightarrow \frac{1}{v} = -\frac{kt}{m} + C \Rightarrow \frac{1}{v} = \frac{kt}{m} - C. \) Since \( v(0) = v_0, \)

\[ C = -\frac{1}{v_0} \quad \text{and} \quad \frac{1}{v} = \frac{kt}{m} + \frac{1}{v_0}. \] Therefore, \( v(t) = \frac{m v_0}{k v_0 t + m} \) \( \frac{ds}{dt} = \frac{mv_0}{k v_0 t + m} \Rightarrow \)

\[ s(t) = \frac{m}{k} \int \frac{k v_0 \, dt}{k v_0 t + m} = \frac{m}{k} \ln|k v_0 t + m| + C'. \] Since \( s(0) = s_0, \) we get \( s_0 = \frac{m}{k} \ln m + C' \Rightarrow \)

\[ C' = s_0 - \frac{m}{k} \ln m \Rightarrow s(t) = s_0 + \frac{m}{k} \left( \ln|k v_0 t + m| - \ln m \right) = s_0 + \frac{m}{k} \ln \left| \frac{k v_0 t + m}{m} \right|. \]

We can rewrite the formulas for \( v(t) \) and \( s(t) \) as \( v(t) = \frac{v_0}{1 + (k v_0/m) t} \) and \( s(t) = s_0 + \frac{m}{k} \ln \left| 1 + \frac{k v_0 t}{m} \right|. \)

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which \( v_0 > 0. \)

Then the term \(-kv^2\) representing the resisting force causes the object to decelerate. The absolute value in the expression for \( s(t) \) is unnecessary (since \( k, v_0, \) and \( m \) are all positive), and \( \lim_{t \to \infty} s(t) = \infty. \) In other words, the object travels infinitely far. However, \( \lim_{t \to \infty} v(t) = 0. \) When \( v_0 < 0, \) the term \(-kv^2\) increases the magnitude of the object’s negative velocity. According to the formula for \( s(t), \) the position of the object approaches \(-\infty\) as \( t \) approaches \( m/k(-v_0): \)

\[ \lim_{t \to \infty} s(t) = -\infty. \] Again the object travels infinitely far, but this time the feat is accomplished in a finite amount of time. Notice also that \( \lim_{t \to -m/(k v_0)} v(t) = -\infty \) when \( v_0 < 0, \) showing that the speed of the object increases without limit.
49. (a) The rate of growth of the area is jointly proportional to \( \sqrt{A(t)} \) and \( M - A(t) \); that is, the rate is proportional to the product of those two quantities. So for some constant \( k \), \( dA/dt = k \sqrt{A} (M - A) \). We are interested in the maximum of the function \( dA/dt \) (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for \( dA/dt \) from the differential equation:

\[
\frac{d}{dt} \left( \frac{dA}{dt} \right) = k \left[ \sqrt{A} \left( -1 \frac{dA}{dt} + (M - A) \cdot \frac{1}{2} A^{-1/2} \frac{dA}{dt} \right) \right] = \frac{1}{2} k A^{-1/2} \frac{dA}{dt} \left[ -2A + (M - A) \right] = \frac{1}{2} k A^{-1/2} \left[ k \sqrt{A} (M - A) \right] [M - 3A] = \frac{1}{2} k^2 (M - A)(M - 3A)
\]

This is 0 when \( M - A = 0 \) [this situation never actually occurs, since the graph of \( A(t) \) is asymptotic to the line \( y = M \), as in the logistic model] and when \( M - 3A = 0 \) \( \Leftrightarrow A(t) = M/3 \). This represents a maximum by the First Derivative Test, since \( \frac{d}{dt} \left( \frac{dA}{dt} \right) \) goes from positive to negative when \( A(t) = M/3 \).

(b) From the CAS, we get \( A(t) = M \left( \frac{C e^\sqrt{M t} - 1}{C e^\sqrt{M t} + 1} \right)^2 \). To get \( C \) in terms of the initial area \( A_0 \) and the maximum area \( M \), we substitute \( t = 0 \) and \( A = A_0 \): \( A_0 = M \left( \frac{C - 1}{C + 1} \right)^2 \) \( \Leftrightarrow \) \( (C + 1) \sqrt{A_0} = (C - 1) \sqrt{M} \) \( \Leftrightarrow \)

\[ C \sqrt{A_0} - \sqrt{A_0} = C \sqrt{M} - \sqrt{M} \Leftrightarrow \sqrt{M} + \sqrt{A_0} = C \sqrt{M} - C \sqrt{A_0} \Leftrightarrow \]

\[ \sqrt{M} + \sqrt{A_0} = C \left( \sqrt{M} - \sqrt{A_0} \right) \Leftrightarrow C = \frac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}} \text{.} \] [Notice that if \( A_0 = 0 \), then \( C = 1 \).]

50. (a) According to the hint we use the Chain Rule: \( m \frac{dv}{dt} = m \frac{dv}{dx} \cdot \frac{dx}{dt} = mv \frac{dv}{dx} = -\frac{mgR^2}{(x + R)^2} \Rightarrow \)

\[
\int v \, dv = \int \frac{-gR^2 \, dx}{(x + R)^2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{x + R} + C. \text{ When } x = 0, v = v_0, \text{ so } \frac{v_0^2}{2} = \frac{gR^2}{0 + R} + C \Rightarrow \]

\[ C = \frac{1}{2} v_0^2 - gR \Rightarrow \frac{1}{2} v^2 - \frac{1}{2} v_0^2 = \frac{gR^2}{x + R} - gR. \text{ Now at the top of its flight, the rocket's velocity will be } 0, \text{ and its height will be } x = h. \text{ Solving for } v_0: -\frac{1}{2} v_0^2 = \frac{gR^2}{h + R} - gR \Rightarrow \frac{v_0^2}{2} = g \left[ -\frac{R^2}{R + h} + \frac{R(h + R)}{R + h} \right] = \frac{gRh}{R + h} \Rightarrow \]

\[ v_0 = \sqrt{\frac{2gRh}{R + h}}. \]

(b) \( v_e = \lim_{h \to \infty} v_0 = \lim_{h \to \infty} \sqrt{\frac{2gRh}{R + h}} = \lim_{h \to \infty} \sqrt{\frac{2gR}{(R/h) + 1}} = \sqrt{2gR} \]

(c) \( g = 9.8 \text{ m/s}^2 = 0.0098 \text{ km/s}^2 \), so \( v_e = \sqrt{2} \cdot 0.0098 \text{ km/s}^2 \cdot 6370 \text{ km} \approx 11.17 \text{ km/s} \).
Chapter 07-Concept Check

1. (a) See Section 7.1, Figure 2 and Equations 7.1.1 and 7.1.2.

(b) Instead of using “top minus bottom” and integrating from left to right, we use “right minus left” and integrate from bottom to top. See Figures 8 and 9 in Section 7.1.

2. The numerical value of the area represents the number of meters by which Sue is ahead of Kathy after 1 minute.

3. (a) See the discussion on pages 371–72.

(b) See the discussion between Examples 5 and 6 in Section 7.2. If the cross-section is a disk, find the radius in terms of x or y and use $A = \pi (\text{radius})^2$. If the cross-section is a washer, find the inner radius $r_{\text{in}}$ and outer radius $r_{\text{out}}$ and use $A = \pi (r_{\text{out}}^2) - \pi (r_{\text{in}}^2)$.

4. (a) $V = 2\pi rh \Delta r = (\text{circumference})(\text{height})(\text{thickness})$

(b) For a typical shell, find the circumference and height in terms of x or y and calculate $V = \int_a^b (\text{circumference})(\text{height})(dx \text{ or } dy)$, where a and b are the limits on x or y.

(c) Sometimes slicing produces washers or disks whose radii are difficult (or impossible) to find explicitly. On other occasions, the cylindrical shell method leads to an easier integral than slicing does.

5. (a) The length of a curve is defined to be the limit of the lengths of the inscribed polygons, as described near Figure 3 in Section 7.4.

(b) See Equation 7.4.2.

(c) See Equation 7.4.4.

6. (a) $S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$

(b) If $x = g(y)$, $c \leq y \leq d$, then $S = \int_c^d 2\pi y \sqrt{1 + [g'(y)]^2} \, dy$.

(c) $S = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2} \, dx$ or $S = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} \, dy$

7. $\int_0^b f(x) \, dx$ represents the amount of work done. Its units are newton-meters, or joules.

8. Let $c(x)$ be the cross-sectional length of the wall (measured parallel to the surface of the fluid) at depth $x$. Then the hydrostatic force against the wall is given by $F = \int_a^b \delta xc(x) \, dx$, where $a$ and $b$ are the lower and upper limits for $x$ at points of the wall and $\delta$ is the weight density of the fluid.

9. (a) The center of mass is the point at which the plate balances horizontally.

(b) See Equations 7.6.12.
10. If a plane region $\mathcal{R}$ that lies entirely on one side of a line $\ell$ in its plane is rotated about $\ell$, then the volume of the resulting solid is the product of the area of $\mathcal{R}$ and the distance traveled by the centroid of $\mathcal{R}$.

11. (a) A differential equation is an equation that contains an unknown function and one or more of its derivatives.

(b) The order of a differential equation is the order of the highest derivative that occurs in the equation.

(c) An initial condition is a condition of the form $y(t_0) = y_0$.

12. See the paragraph preceding Example 6 in Section 7.7.

13. A separable equation is a first-order differential equation in which the expression for $dy/dx$ can be factored as a function of $x$ times a function of $y$, that is, $dy/dx = g(x)f(y)$. We can solve the equation by integrating both sides of the equation $dy/f(y) = g(x)dx$ and solving for $y$. 
1. The curves intersect when \( x^2 = 4x - x^2 \) \( \iff \) \( 2x^2 - 4x = 0 \) \( \iff \) 
\( 2x(x - 2) = 0 \) \( \iff \) \( x = 0 \) or 2.

\[
A = \int_0^2 [(4x - x^2) - x^2] \, dx = \int_0^2 (4x - 2x^2) \, dx
\]
\[
= \left[ 2x^2 - \frac{2}{3}x^3 \right]_0^2 = \left[ (8 - \frac{16}{3}) - 0 \right] = \frac{8}{3}
\]

2. The curves \( y = 1/x \) and \( y = x^2 \) intersect when \( 1/x = x^2 \) \( \iff \) \( x^3 = 1 \) \( \iff \) \( x = 1 \).

\[
A = \int_0^1 (x^2 - 0) \, dx + \int_1^e \left( \frac{1}{x} - 0 \right) \, dx = \left[ \frac{1}{3}x^3 \right]_0^1 + \left[ \ln|x| \right]_1^e
\]
\[
= \left( \frac{1}{3} - 0 \right) + (1 - 0) = \frac{4}{3}
\]

3. If \( x \geq 0 \), then \( |x| = x \), and the graphs intersect when \( x - 2x^2 = 0 \) \( \iff \) \( 2x^2 + x - 1 = 0 \) \( \iff \) \( (2x - 1)(x + 1) = 0 \) \( \iff \) \( x = \frac{1}{2} \) or \(-1\), but \(-1 < 0\). By symmetry, we can double the area from \( x = 0 \) to \( x = \frac{1}{2} \).

\[
A = 2 \int_0^{1/2} \left[ (1 - 2x^2) - x \right] \, dx = 2 \int_0^{1/2} (-2x^2 - x + 1) \, dx
\]
\[
= 2\left[ -\frac{2}{3}x^3 - \frac{1}{2}x^2 + x \right]_0^{1/2} = 2\left[ \left( -\frac{1}{12} - \frac{1}{8} + \frac{1}{2} \right) - 0 \right]
\]
\[
= 2\left( \frac{7}{12} \right) = \frac{7}{6}
\]

4. \( y^2 + 3y = -y \) \( \iff \) \( y^2 + 4y = 0 \) \( \iff \) \( y(y + 4) = 0 \) \( \iff \) \( y = 0 \) or \(-4\).

\[
A = \int_{-4}^0 [-y - (y^2 + 3y)] \, dy + \int_{-4}^0 (-y^2 - 4y) \, dy
\]
\[
= \left[ -\frac{1}{3}y^3 - 2y^2 \right]_{-4}^0 = 0 - \left( \frac{64}{3} - 32 \right) = \frac{32}{3}
\]

5. Using washers with inner radius \( x^2 \) and outer radius \( 2x \), we have

\[
V = \pi \int_0^2 [(2x)^2 - (x^2)^2] \, dx = \pi \int_0^2 (4x^2 - x^4) \, dx
\]
\[
= \pi \left[ \frac{3}{2}x^2 - \frac{1}{8}x^8 \right]_0^2 = \pi \left( \frac{23}{3} - \frac{256}{8} \right)
\]
\[
= 32\pi \cdot \frac{1}{15} = \frac{64}{15}\pi
\]
6. \(1 + y^2 = y + 3 \iff y^2 - y - 2 = 0 \iff (y - 2)(y + 1) = 0 \iff y = 2 \text{ or } -1.
\[
V = \pi \int_{-1}^{2} \left[ (y + 3)^2 - (1 + y^2)^2 \right] dy = \pi \int_{-1}^{2} (y^2 + 6y + 9 - 1 - 2y^2 - y^4) dy
\]
\[= \pi \int_{-1}^{2} (8 + 6y - y^2 - y^4) dy = \pi \left[ 8y + 3y^2 - \frac{1}{2}y^3 - \frac{1}{5}y^5 \right]_{-1}^{2}
\]
\[= \pi \left[ (16 + 12 - \frac{8}{5} - \frac{32}{5}) - (-8 + 3 + \frac{1}{2} + \frac{1}{5}) \right] = \pi \left( 33 - \frac{32}{5} \right) = \frac{117}{5} \pi.
\]

7. \(V = \pi \int_{-2}^{3} \left[ (9 - y^2)^2 - (-1 - y^2)^2 \right] dy
\]
\[= 2\pi \int_{0}^{2} \left[ (10 - y^2)^2 - (10 - 2y)^2 \right] dy = 2\pi \int_{0}^{2} (100 - 20y^2 + y^4 - 1 - 20y + 4y^2) dy
\]
\[= 2\pi \int_{0}^{2} (99 - 16y^2 + y^4) dy = 2\pi \left[ 99y - \frac{16}{2}y^3 + \frac{1}{5}y^5 \right]_{0}^{2}
\]
\[= 2\pi (97 - 180 + \frac{247}{5}) = \frac{1658}{5} \pi.
\]

8. \(V = \pi \int_{-2}^{2} \left[ (9 - x^2)^2 - (x^2 + 1)^2 \right] dx
\]
\[= \pi \int_{-2}^{2} \left[ (10 - x^2)^2 - (x^2 + 2)^2 \right] dx
\]
\[= 2\pi \int_{0}^{2} (96 - 24x^2) dx = 48\pi \left[ 4x - \frac{x^3}{6} \right]_{0}^{2}
\]
\[= 48\pi \left( 8 - \frac{8}{3} \right) = 256\pi.
\]

9. The graph of \(x^2 - y^2 = a^2\) is a hyperbola with right and left branches.

Solving for \(y\) gives \(y^2 = x^2 - a^2 \iff y = \pm \sqrt{x^2 - a^2}.

We’ll use shells and the height of each shell is
\(\sqrt{x^2 - a^2} - (-\sqrt{x^2 - a^2}) = 2 \sqrt{x^2 - a^2}.

The volume is \(V = \int_{a}^{a+h} 2\pi x \cdot 2 \sqrt{x^2 - a^2} dx\). To evaluate, let \(u = x^2 - a^2\),
so \(du = 2x dx\) and \(x dx = \frac{1}{2} du\). When \(x = a, u = 0\), and when \(x = a + h, u = (a + h)^2 - a^2 = a^2 + 2ah + h^2 - a^2 = 2ah + h^2\).

Thus, \(V = 4\pi \int_{0}^{2ah+h^2} \sqrt{u} \left( \frac{1}{2} du \right) = 2\pi \left[ \frac{2}{3} u^{3/2} \right]_{0}^{2ah+h^2} = \frac{4}{3} \pi (2ah + h^2)^{3/2}.\)
10. A shell has radius $x$, circumference $2\pi x$, and height $\tan x - x$.

$$V = \int_0^{\pi/3} 2\pi x (\tan x - x) \, dx$$

11. A shell has radius $\frac{\pi}{2} - x$, circumference $2\pi \left(\frac{\pi}{2} - x\right)$, and height $\cos^2 x - \frac{1}{4}$.

$$y = \cos^2 x \text{ intersects } y = \frac{1}{4} \text{ when } \cos^2 x = \frac{1}{4} \iff \cos x = \pm \frac{1}{2} \quad \quad [|x| \leq \pi/2] \iff x = \pm \frac{\pi}{3}.$$  

$$V = \int_{-\pi/2}^{\pi/3} 2\pi \left(\frac{\pi}{2} - x\right) \left(\cos^2 x - \frac{1}{4}\right) \, dx$$

12. A washer has outer radius $2 - x^2$ and inner radius $2 - \sqrt{x}$.

$$V = \int_0^1 \pi \left[(2 - x^2)^2 - (2 - \sqrt{x})^2\right] \, dx$$

13. (a) A cross-section is a washer with inner radius $x^2$ and outer radius $x$.

$$V = \int_0^1 \pi \left[(x)^2 - (x^2)^2\right] \, dx = \int_0^1 \pi (x^2 - x^4) \, dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{6}x^6\right]_0^1 = \pi \left[\frac{1}{3} - \frac{1}{6}\right] = \frac{1}{6} \pi$$

(b) A cross-section is a washer with inner radius $y$ and outer radius $\sqrt{y}$.

$$V = \int_0^1 \pi \left[(\sqrt{y})^2 - y^2\right] \, dy = \int_0^1 \pi (y - y^2) \, dy = \pi \left[\frac{1}{3}y^2 - \frac{1}{2}y^3\right]_0^1 = \pi \left[\frac{1}{3} - \frac{1}{2}\right] = \frac{\pi}{6}$$

(c) A cross-section is a washer with inner radius $2 - x$ and outer radius $2 - x^2$.

$$V = \int_0^1 \pi \left[(2 - x^2)^2 - (2 - x)^2\right] \, dx = \int_0^1 \pi (x^4 - 5x^2 + 4x) \, dx = \pi \left[\frac{1}{6}x^6 - \frac{5}{2}x^2 + 2x^2\right]_0^1 = \pi \left[\frac{1}{6} - \frac{5}{2} + 2\right] = \frac{8}{18} \pi$$
14. (a) \[ A = \int_{\frac{1}{2}}^{1} (2x - x^2 - x^3) \, dx = \left[ x^2 - \frac{1}{3} x^3 - \frac{1}{4} x^4 \right]_{\frac{1}{2}}^{1} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{5}{12} \]

(b) A cross-section is a washer with inner radius \( x^2 \) and outer radius \( 2x - x^2 \), so its area is \( \pi (2x - x^2)^2 - \pi (x^2)^2 \).

\[
V = \int_{\frac{1}{2}}^{1} A(x) \, dx = \int_{\frac{1}{2}}^{1} \pi [(2x - x^2)^2 - (x^2)^2] \, dx = \int_{\frac{1}{2}}^{1} \pi (4x^2 - 4x^4 + x^4 - x^6) \, dx
= \pi \left[ \frac{3}{2} x^2 - x^4 + \frac{1}{6} x^5 - \frac{1}{3} x^7 \right]_{\frac{1}{2}}^{1} = \pi \left( \frac{3}{2} - 1 + \frac{1}{6} - \frac{1}{3} \right) = \frac{41}{90} \pi
\]

(c) Using the method of cylindrical shells,

\[
V = \int_{0}^{1} 2\pi x (2x - x^2 - x^3) \, dx = \int_{0}^{1} 2\pi (2x^2 - x^2 - x^4) \, dx = 2\pi \left[ \frac{3}{2} x^2 - \frac{1}{4} x^4 - \frac{1}{6} x^6 \right]_{0}^{1} = 2\pi \left( \frac{3}{2} - \frac{1}{4} - \frac{1}{6} \right) = \frac{13}{36} \pi
\]

15. (a) Using the Midpoint Rule on \([0, 1]\) with \( f(x) = \tan(x^2) \) and \( n = 4 \), we estimate

\[ A = \int_{0}^{1} \tan(x^2) \, dx \approx \frac{1}{4} \left[ \tan \left( \frac{1}{4} \right)^2 + \tan \left( \frac{3}{8} \right)^2 + \tan \left( \frac{7}{8} \right)^2 + \tan \left( \frac{15}{8} \right)^2 \right] \approx \frac{1}{4} (1.53) \approx 0.38 \]

(b) Using the Midpoint Rule on \([0, 1]\) with \( f(x) = \pi \tan^2(x^2) \) (for disks) and \( n = 4 \), we estimate

\[ V = \int_{0}^{1} f(x) \, dx \approx \frac{1}{4} \pi \left[ \tan^2 \left( \frac{1}{8} \right)^2 + \tan^2 \left( \frac{3}{8} \right)^2 + \tan^2 \left( \frac{7}{8} \right)^2 + \tan^2 \left( \frac{15}{8} \right)^2 \right] \approx \frac{3}{4} (1.114) \approx 0.87 \]

16. (a) From the graph, we see that the curves intersect at \( x = 0 \) and at \( x = \alpha \approx 0.75 \), with \( 1 - x^2 > x^6 - x + 1 \) on \((0, \alpha)\).

(b) The area of \( \mathcal{R} \) is \( A = \int_{0}^{\alpha} [(1 - x^2) - (x^6 - x + 1)] \, dx = \left[ -\frac{1}{3} x^3 - \frac{1}{7} x^7 + \frac{1}{4} x^4 \right]_{0}^{\alpha} \approx 0.12 \).

(c) Using washers, the volume generated when \( \mathcal{R} \) is rotated about the \( x \)-axis is

\[ V = \pi \int_{0}^{\alpha} [(1 - x^2)^2 - (x^6 - x + 1)^2] \, dx = \pi \int_{0}^{\alpha} (-x^2 + 2x^4 - 2x^2 - 2x^6 + x^2^3 + 2x) \, dx
= \pi \left[ -\frac{1}{12} x^3 + \frac{1}{4} x^6 - \frac{3}{2} x^7 + \frac{1}{5} x^8 - x^9 + x^2 \right]_{0}^{\alpha} \approx 0.54 \]

(d) Using shells, the volume generated when \( \mathcal{R} \) is rotated about the \( y \)-axis is

\[ V = \int_{0}^{\alpha} 2\pi x [(1 - x^2) - (x^6 - x + 1)] \, dx = 2\pi \int_{0}^{\alpha} (-x^2 - x^7 + x^2) \, dx = 2\pi \left[ -\frac{1}{4} x^4 - \frac{1}{8} x^8 + \frac{1}{8} x^2 \right]_{0}^{\alpha} \approx 0.31 \]

17. \[ \int_{0}^{\pi/2} 2\pi x \cos x \, dx = \int_{0}^{\pi/2} (2\pi x) \cos x \, dx \]

The solid is obtained by rotating the region \( \mathcal{R} = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x\} \) about the \( y \)-axis.

18. \[ \int_{0}^{\pi/2} 2\pi \cos^2 x \, dx = \int_{0}^{\pi/2} \pi (\sqrt{2} \cos x)^2 \, dx \]

The solid is obtained by rotating the region \( \mathcal{R} = \{(x, y) \mid 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \sqrt{2} \cos x\} \) about the \( x \)-axis.
19. \( \int_0^\pi \pi (2 - \sin x)^2 \, dx \)

The solid is obtained by rotating the region \( R = \{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq 2 - \sin x\} \) about the \( x \)-axis.

20. \( \int_r^x 2\pi(y - y^2) \, dy \)

The solid is obtained by rotating the region \( R = \{(x, y) \mid 0 \leq x \leq 4y - y^2, 0 \leq y \leq 4\} \) about the line \( y = 6 \).

21. Take the base to be the disk \( x^2 + y^2 \leq 9 \). Then \( V = \int_0^6 A(x) \, dx \), where \( A(x) \) is the area of the isosceles right triangle whose hypotenuse lies along the line \( x = x_0 \) in the \( xy \)-plane. The length of the hypotenuse is \( 2\sqrt{9 - x^2} \) and the length of each leg is \( \sqrt{9 - x^2} \). \( A(x) = \frac{1}{2} \left( \sqrt{9 - x^2} \right)^2 = 9 - x^2 \), so

\[
V = 2 \int_0^3 A(x) \, dx = 2 \int_0^3 (9 - x^2) \, dx = 2 \left[ 9x - \frac{1}{3} x^3 \right]_0^3 = 2(27 - 9) = 36
\]

22. \( V = \int_{-1}^1 A(x) \, dx = 2 \int_0^1 A(x) \, dx = 2 \int_0^1 [(2 - x^3) - x^3] \, dx = 2 \int_0^1 [2(1 - x^3)] \, dx \\
= 8 \int_0^1 (1 - 2x^2 + x^4) \, dx = 8 \left[ x - \frac{2}{3} x^3 + \frac{1}{5} x^5 \right]_0^1 = 8 \left( 1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{64}{15}
\]

23. Equilateral triangles with sides measuring \( \frac{1}{2} x \) meters have height \( \frac{1}{2} x \sin 60^\circ = \frac{x\sqrt{3}}{8} \). Therefore,

\( A(x) = \frac{1}{2} \cdot \frac{1}{2} x \cdot \frac{x\sqrt{2}}{8} = \frac{\sqrt{3}}{64} x^2 \). \( V = \int_0^{20} A(x) \, dx = \frac{\sqrt{3}}{64} \int_0^{20} x^2 \, dx = \frac{\sqrt{3}}{64} \left[ \frac{1}{3} x^3 \right]_0^{20} = \frac{5008 \sqrt{3}}{64} = \frac{128 \sqrt{3}}{3} \text{ m}^3. \)

24. (a) By the symmetry of the problem, we consider only the solid to the right of the origin. The semicircular cross-sections perpendicular to the \( x \)-axis have radius \( 1 - x \), so \( A(x) = \frac{1}{2} \pi (1 - x)^2 \). Now we can calculate

\[
V = 2 \int_0^1 A(x) \, dx = 2 \int_0^1 \frac{1}{2} \pi (1 - x)^2 \, dx = \int_0^1 \pi (1 - x)^2 \, dx = -\frac{\pi}{3} \left[ 1 - x \right]^1_0 = \frac{\pi}{3}
\]

(b) Cut the solid with a plane perpendicular to the \( x \)-axis and passing through the \( y \)-axis. Fold the half of the solid in the region \( x \leq 0 \) under the \( xy \)-plane so that the point \((-1, 0)\) comes around and touches the point \((1, 0)\). The resulting solid is a right circular cone of radius 1 with vertex at \((x, y, z) = (1, 0, 0)\) and with its base in the \( yz \)-plane, centered at the origin.

The volume of this cone is \( \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \cdot 1^2 \cdot 1 = \frac{\pi}{3} \).

25. \( y = \frac{1}{8} (x^2 + 4)^{9/2} \Rightarrow \frac{dy}{dx} = \frac{1}{4} (x^2 + 4)^{1/2} (2x) \Rightarrow \)

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( \frac{1}{4} x(x^2 + 4)^{1/2} \right)^2 = 1 + \frac{1}{16} x^2(x^2 + 4) = \frac{1}{16} x^4 + x^2 + 1 = \left( \frac{1}{2} x^2 + 1 \right)^2.
\]

Thus, \( L = \int_0^2 \sqrt{\left( \frac{1}{2} x^2 + 1 \right)^2} \, dx = \int_0^3 \left( \frac{1}{2} x^2 + 1 \right) \, dx = \left[ \frac{1}{2} x^3 + x \right]_0^3 = \frac{19}{2}. \)
26. \( y = 2 \ln(\sin \frac{1}{2}x) \Rightarrow \frac{dy}{dx} = 2 \frac{1}{\sin(\frac{1}{2}x)} \cdot \cos(\frac{1}{2}x) \cdot \frac{1}{2} = \cot(\frac{1}{2}x) \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cot^2(\frac{1}{2}x) = \csc^2(\frac{1}{2}x). \)

Thus,

\[
L = \int_{\pi/3}^{\pi} \sqrt{\csc^2(\frac{1}{2}x)} \, dx = \int_{\pi/3}^{\pi} \csc(\frac{1}{2}x) \, dx = \int_{\pi/6}^{\pi/3} \csc u \, (2 \, du) \quad \left[ u = \frac{1}{2}x, \quad du = \frac{1}{2} \, dx \right]
\]

\[
= 2 \left[ \ln|\csc u - \cot u| \right]_{\pi/6}^{\pi/3} = 2 \left[ \ln|\csc \frac{\pi}{3} - \cot \frac{\pi}{3}| - \ln|\csc \frac{\pi}{6} - \cot \frac{\pi}{6}| \right]
\]

\[
= 2 \left[ \ln |1 - 0| - \ln |2 - \sqrt{3}| \right] = -2 \ln(2 - \sqrt{3}) \approx 2.63
\]

27. (a) \( y = \frac{x^4}{16} + \frac{1}{2x^2} = \frac{1}{16}x^4 + \frac{1}{2}x^{-2} \Rightarrow \frac{dy}{dx} = \frac{1}{4}x^3 - x^{-3} \Rightarrow
\]

\[1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \frac{1}{16}x^6 - \frac{1}{2}x^{-6} = \frac{1}{16}x^6 + \frac{1}{2}x^{-6} = \left(\frac{1}{4}x^2 + x^{-2}\right)^2.\]

Thus, \( L = f_1^2 \left(\frac{1}{4}x^2 + x^{-2}\right) \, dx = \frac{1}{16}x^4 - \frac{1}{2}x^{-2} \right|_1 = (1 - \frac{1}{8}) - (\frac{1}{16} - \frac{1}{2}) = \frac{21}{16}.\)

(b) \( S = \int_1^2 2\pi x \left(\frac{1}{4}x^2 + x^{-2}\right) \, dx = 2\pi \int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right) \, dx = 2\pi \left[\frac{1}{20}x^5 - \frac{1}{2}\right]_1
\]

\[= 2\pi \left[\frac{21}{20} - \frac{1}{2}\right] = 2\pi \left(\frac{21}{20} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi \left(\frac{11}{20}\right) = \frac{11}{10} \pi.\]

28. (a) \( y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2 \Rightarrow
\]

\[
S = \int_0^1 2\pi x \sqrt{1 + 4x^2} \, dx = \int_1^5 \frac{\pi}{4} \sqrt{u} \, du \quad \left[ u = 1 + 4x^2 \right] = \frac{\pi}{6} \left[u^{3/2}\right]_1 = \frac{\pi}{6}(5^{3/2} - 1)
\]

(b) \( y = x^2 \Rightarrow 1 + (y')^2 = 1 + 4x^2. \) So

\[
S = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} \, dx = 2\pi \int_0^1 \frac{1}{2}u^{3/2} \, du \quad \left[ u = 2x \right] = \frac{\pi}{4} \int_0^2 u^{3/2} \sqrt{1 + u^2} \, du
\]

\[= \frac{\pi}{4} \left[u(1 + 2u^2) \sqrt{1 + u^2} - \frac{1}{3} \ln(1 + u^2) \right]_0^n \quad \left[ u = \tan \theta \text{ or use Formula } 22 \right]
\]

\[= \frac{\pi}{4} \left[\frac{1}{3}9\sqrt{5} - \frac{1}{3} \ln(2 + \sqrt{5}) - 0 \right] = \frac{1}{4} \left[18\sqrt{5} - \ln(2 + \sqrt{5})\right]
\]

29. \( y = \sin x \Rightarrow y' = \cos x \Rightarrow 1 + (y')^2 = 1 + \cos^2 x. \) Let \( f(x) = \sqrt{1 + \cos^2 x}. \) Then

\[
L = \int_0^\pi f(x) \, dx \approx S_{10}
\]

\[= \frac{(\pi - 0)/10}{3} \left[f(0) + 4f(\frac{\pi}{10}) + 2f(\frac{2\pi}{10}) + 4f(\frac{3\pi}{10}) + 2f(\frac{4\pi}{10}) + 4f(\frac{5\pi}{10}) + 2f(\frac{6\pi}{10}) + 4f(\frac{7\pi}{10}) + 2f(\frac{8\pi}{10}) + 4f(\frac{9\pi}{10}) + f(\pi)\right]
\]

\[\approx 3.820188\]
30. \( S = \int_{0}^{\pi} 2\pi y \, ds = \int_{0}^{\pi} 2\pi \sin x \sqrt{1 + \cos^2 x} \, dx \). Let \( g(x) = 2\pi \sin x \sqrt{1 + \cos^2 x} \). Then
\[
S = \int_{0}^{\pi} g(x) \, dx = S_{10}
\]
\[
= \frac{(\pi - 9)^{1/10}}{3} \left[ g(0) + 4g(\frac{\pi}{10}) + 2g(\frac{2\pi}{10}) + 4g(\frac{3\pi}{10}) + 2g(\frac{4\pi}{10}) + 4g(\frac{5\pi}{10}) + 2g(\frac{6\pi}{10}) + 4g(\frac{7\pi}{10}) + 2g(\frac{8\pi}{10}) + 4g(\frac{9\pi}{10}) + g(\pi) \right]
\]
\approx 14.426045

31. \( y = \int_{1}^{\pi} \sqrt{x - 1} - 1 \, dt \quad \Rightarrow \quad dy/dx = \sqrt{x - 1} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \left( \sqrt{x - 1} \right) = \sqrt{x}.

Thus, \( L = \int_{1}^{16} \sqrt{x} \, dx = \int_{1}^{16} x^{1/4} \, dx = \frac{4}{5} \left[ x^{5/4} \right]_{1}^{16} = \frac{4}{5} (32 - 1) = \frac{124}{5} \).

32. \( S = \int_{1}^{16} 2\pi x \, ds = 2\pi \int_{1}^{16} x \cdot x^{1/4} \, dx = 2\pi \int_{1}^{16} x^{5/4} \, dx = 2\pi \cdot \frac{8}{5} \left[ x^{9/4} \right]_{1}^{16} = \frac{8}{5} (512 - 1) = \frac{4068}{5} \pi \)

33. \( f(x) = kx \quad \Rightarrow \quad 30 \text{ N} = k(15 - 12) \text{ cm} \quad \Rightarrow \quad k = 10 \text{ N/cm} = 1000 \text{ N/m}. \quad 20 \text{ cm} - 12 \text{ cm} = 0.08 \text{ m} \Rightarrow \)
\[
W = \int_{0}^{0.08} kx \, dx = 1000 \int_{0}^{0.08} x \, dx = 500 [x^2]_{0}^{0.08} = 500(0.08)^2 = 3.2 \text{ N-m} = 3.2 \text{ J}.
\]

34. The work needed to raise the elevator alone is 1600 lb \times 30 ft = 48,000 \text{ ft-lb}. The work needed to raise the bottom 170 ft of cable is 170 ft \times 10 \text{ lb/ft} \times 30 \text{ ft} = 51,000 \text{ ft-lb}. The work needed to raise the top 30 ft of cable is \( \int_{0}^{30} 10x \, dx = \left[ 5x^2 \right]_{0}^{30} = 5 \cdot 900 = 4500 \text{ ft-lb} \). Adding these, we see that the total work needed is 48,000 + 51,000 + 4,500 = 103,500 \text{ ft-lb}.

35. (a) The parabola has equation \( y = ax^2 \) with vertex at the origin and passing through \( (4, 4) \).
\[
4 = a \cdot 4^2 \quad \Rightarrow \quad a = \frac{1}{4} \quad \Rightarrow \quad y = \frac{1}{4}x^2 \quad \Rightarrow \quad x^2 = 4y \quad \Rightarrow \quad x = 2\sqrt{y}.
\]
Each circular disk has radius \( 2\sqrt{y} \) and is moved \( 4 - y \) ft.
\[
W = \int_{0}^{4} \pi \left( 2\sqrt{y} \right)^2 62.5(4 - y) \, dy = 250\pi \int_{0}^{4} y(4 - y) \, dy
\]
\[
= 250\pi \left[ \frac{2y^2}{2} - \frac{1}{2}y^3 \right]_{0}^{4} = 250\pi (32 - \frac{64}{3}) = \frac{8000\pi}{3} \approx 8378 \text{ ft-lb}
\]

(b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level)—call it \( h \)—unknown. \( W = 4000 \quad \Rightarrow \quad 250\pi \left[ 2y^2 - \frac{1}{2}y^3 \right]_{h}^{4} = 4000 \quad \Rightarrow \quad \frac{18}{\pi} = \left[ (32 - \frac{64}{3}) - (2h^2 - \frac{1}{2}h^3) \right] \quad \Rightarrow \quad h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0. \) We graph the function \( f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi} \) on the interval \([0, 4]\) to see where it is 0. From the graph, \( f(h) = 0 \) for \( h \approx 2.1 \).
So the depth of water remaining is about 2.1 ft.
36. \( F = \int_0^4 \delta(4-y)2(2\sqrt{y}) \, dy = 4\delta \int_0^4 (4y^{1/2} - y^{3/2}) \, dy \)
\[ = 4\delta \left[ \frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5} \right]_0^4 = 4\delta \left( \frac{64}{9} - \frac{64}{5} \right) = 256\delta \left( \frac{1}{9} - \frac{1}{5} \right) \]
\[ = \frac{64\delta}{16} \approx 2133.3 \text{ lb} \quad [\delta \approx 62.5 \text{ lb/ft}^2] \]

37. As in Example 5 of Section 7.6, \( \frac{\alpha}{1-x} = \frac{1}{3} \Rightarrow 2a = 1 - x \text{ and } w = 2(0.5 + a) = 1 + 2a = 1 + 1 - x = 2 - x. \)

Thus, \( F = \int_0^1 \rho g x(2 - x) \, dx = \rho g \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \rho g \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3} \rho g = \frac{1}{3}(1000)(9.8) \approx 6533 \text{ N.} \)

38. An equation of the line passing through \((0, 0)\) and \((3, 2)\) is \( y = \frac{2}{3}x \). \( A = \frac{1}{2} \cdot 3 = 3 \). Therefore, using Equations 7.5.12,
\( \bar{x} = \frac{1}{3} \int_0^3 x \left( \frac{2}{3}x \right) \, dx = \frac{2}{9} \left[ x^3 \right]_0^3 = 2 \) and \( \bar{y} = \frac{1}{3} \int_0^3 \left( \frac{2}{3}x \right)^2 \, dx = \frac{2}{27} \left[ x^3 \right]_0^3 = \frac{2}{3} \). Thus, the centroid is \((\bar{x}, \bar{y}) = (2, \frac{2}{3})\).

39. \( A = \int_0^4 \left( \sqrt{x} - \frac{1}{3}x \right) \, dx = \left[ \frac{2}{3}x^{3/2} - \frac{1}{8}x^2 \right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3} \)

\[ \bar{x} = \frac{1}{A} \int_0^4 x \left( \sqrt{x} - \frac{1}{3}x \right) \, dx = \frac{3}{4} \int_0^4 \left( x^{3/2} - \frac{1}{8}x^2 \right) \, dx \]
\[ = \frac{3}{4} \left[ \frac{2}{5}x^{5/2} - \frac{1}{3}x^3 \right]_0^4 = \frac{3}{5} \left( \frac{64}{5} - \frac{64}{3} \right) = \frac{3}{5} \left( \frac{64}{5} - \frac{64}{3} \right) = \frac{8}{3} \]

\[ \bar{y} = \frac{1}{A} \int_0^4 \frac{1}{2} \left[ \left( \sqrt{x} \right)^2 - \left( \frac{1}{3}x \right)^2 \right] \, dx = \frac{3}{8} \int_0^4 \frac{1}{2} \left( x - \frac{1}{3}x^2 \right) \, dx = \frac{3}{8} \left[ \frac{1}{2}x^2 - \frac{1}{12}x^3 \right]_0^4 = \frac{3}{8} \left( 8 - \frac{16}{3} \right) = \frac{3}{8} \left( \frac{8}{3} \right) = 1 \]

Thus, the centroid is \((\bar{x}, \bar{y}) = (\frac{8}{3}, 1)\).

40. From the symmetry of the region, \( \bar{x} = \frac{\pi}{2} \). \( A = \int_{\pi/4}^{3\pi/4} \sin x \, dx = \left[ -\cos x \right]_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} - \left( -\frac{\sqrt{2}}{2} \right) = \sqrt{2} \)

\[ \bar{y} = \frac{1}{A} \int_{\pi/4}^{3\pi/4} \frac{1}{2} \sin^2 x \, dx = \frac{1}{A} \int_{\pi/4}^{3\pi/4} \frac{1}{4} (1 - \cos 2x) \, dx \]
\[ = \frac{1}{4\sqrt{2}} \left[ x - \frac{1}{2} \sin 2x \right]_{\pi/4}^{3\pi/4} \]
\[ = \frac{1}{4\sqrt{2}} \left[ \frac{3\pi}{4} - \frac{1}{2} \left( -1 \right) - \frac{\pi}{4} + \frac{1}{2} \cdot 1 \right] = \frac{1}{4\sqrt{2}} \left( \frac{3\pi}{4} + 1 \right) \]

Thus, the centroid is \((\bar{x}, \bar{y}) = \left( \frac{\pi}{2}, \frac{1}{4\sqrt{2}} \left( \frac{3\pi}{4} + 1 \right) \right) \approx (1.57, 0.45)\).

41. The centroid of this circle, \((1, 0)\), travels a distance \(2\pi(1)\) when the lamina is rotated about the \(y\)-axis. The area of the circle is \(\pi(1)^2\). So by the Theorem of Pappus, \( V = A(2\pi \bar{x}) = \pi(1)^2 2\pi(1) = 2\pi^2 \).
42. The semicircular region has an area of \( \frac{1}{2} \pi r^2 \), and sweeps out a sphere of radius \( r \) when rotated about the \( x \)-axis.

\[ x = 0 \] because of symmetry about the line \( x = 0 \). And by the Theorem of Pappus, \( V = A(2\pi y) \) \( \Rightarrow \)

\[ \frac{4}{3} \pi r^2 = \frac{1}{2} \pi r^2 (2\pi y) \Rightarrow y = \frac{2}{3} r. \] Thus, the centroid is \( (x, y) = (0, \frac{2}{3} r) \).

43. \( 2ye^{y^2} y' = 2x + 3\sqrt{x} \Rightarrow 2ye^{y^2} \frac{dy}{dx} = 2x + 3\sqrt{x} \Rightarrow 2ye^{y^2} dy = \left(2x + 3\sqrt{x}\right) dx \Rightarrow \\
\int 2ye^{y^2} dy = \int \left(2x + 3\sqrt{x}\right) dx \Rightarrow e^{y^2} = x^2 + 2x^{3/2} + C \Rightarrow y^2 = \ln(x^2 + 2x^{3/2} + C) \Rightarrow \\
y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}

44. \( \frac{dx}{dt} = 1 - t + x - tx = 1(1 - t) + x(1 - t) = (1 + x)(1 - t) \Rightarrow \frac{dx}{1 + x} = (1 - t) dt \Rightarrow \\
\int \frac{dx}{1 + x} = \int (1 - t) dt \Rightarrow \ln|1 + x| = t - \frac{1}{2} t^2 + C \Rightarrow |1 + x| = e^{t - \frac{1}{2} t^2 + C} \Rightarrow \\
1 + x = \pm e^{t - \frac{1}{2} t^2} \cdot e^C \Rightarrow x = -1 + Ke^{t - \frac{1}{2} t^2} \), where \( K \) is any nonzero constant.

45. \( \frac{dr}{dt} + 2tr = r \Rightarrow \frac{dr}{dt} = r - 2tr = r(1 - 2t) \Rightarrow \int \frac{dr}{r} = \int (1 - 2t) dt \Rightarrow \ln|r| = t - t^2 + C \Rightarrow \\
|r| = e^{t - t^2 + C} = ke^{t - t^2}. \) Since \( r(0) = 5 \), \( 5 = ke^0 = k. \) Thus, \( r(t) = 5e^{t - t^2} \).

46. \( (1 + \cos x)y' = (1 + e^{-y}) \sin x \Rightarrow \frac{dy}{1 + e^{-y}} = \frac{\sin x \, dx}{1 + \cos x} \Rightarrow \int \frac{dy}{1 + e^{-y}} = \int \frac{\sin x \, dx}{1 + \cos x} \Rightarrow \\
\int \frac{e^y \, dy}{1 + e^y} = \int \frac{\sin x \, dx}{1 + \cos x} \Rightarrow \ln|1 + e^y| = -\ln|1 + \cos x| + C \Rightarrow \ln(1 + e^y) = -\ln(1 + \cos x) + C \Rightarrow \\
1 + e^y = e^{-\ln(1 + \cos x)} \cdot e^C \Rightarrow e^y = ke^{-\ln(1 + \cos x)} - 1 \Rightarrow y = \ln[ke^{-\ln(1 + \cos x)} - 1]. \) Since \( y(0) = 0 \), \( 0 = \ln[ke^{-\ln 2} - 1] \Rightarrow e^0 = k(\frac{1}{2}) - 1 \Rightarrow k = 4. \) Thus, \( y(x) = \ln[4e^{-\ln(1 + \cos x)} - 1]. \) An equivalent form is \( y(x) = \ln \left(\frac{3 - \cos x}{1 + \cos x}\right). \)
47. (a) We sketch the direction field and four solution curves, as shown. Note that the slope \( y' = x/y \) is not defined on the line \( y = 0 \).

(b) \( y' = x/y \) \( \iff \) \( y \, dy = x \, dx \) \( \iff \) \( y^2 = x^2 + C \). For \( C = 0 \), this is the pair of lines \( y = \pm x \). For \( C \neq 0 \), it is the hyperbola \( x^2 - y^2 = -C \).

48. (a) \( \mathcal{R}_1 \) is the region below the graph of \( y = x^2 \) and above the \( x \)-axis between \( x = 0 \) and \( x = b \), and \( \mathcal{R}_2 \) is the region to the left of the graph of \( x = \sqrt{y} \) and to the right of the \( y \)-axis between \( y = 0 \) and \( y = b^2 \). So the area of \( \mathcal{R}_1 \) is
\[
A_1 = \int_0^b x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_0^b = \frac{1}{3} b^3,
\]
and the area of \( \mathcal{R}_2 \) is
\[
A_2 = \int_0^{b^2} \sqrt{y} \, dy = \left[ \frac{2}{3} y^{3/2} \right]_0^{b^2} = \frac{2}{3} b^3.
\]
So there is no solution to \( A_1 = A_2 \) for \( b \neq 0 \).

(b) Using disks, we calculate the volume of rotation of \( \mathcal{R}_1 \) about the \( x \)-axis to be \( V_{1,x} = \pi \int_0^b (x^2)^2 \, dx = \frac{1}{5} \pi b^5 \).

Using cylindrical shells, we calculate the volume of rotation of \( \mathcal{R}_1 \) about the \( y \)-axis to be
\[
V_{1,y} = 2 \pi \int_0^b x \,(x^2) \, dx = 2 \pi \left[ \frac{1}{4} x^4 \right]_0^b = \frac{1}{2} \pi b^4.
\]
So \( V_{1,x} = V_{1,y} \) \( \iff \) \( \frac{1}{5} \pi b^5 = \frac{1}{2} \pi b^4 \) \( \iff \) \( 2b = 5 \) \( \iff \) \( b = \frac{5}{2} \).

So the volumes of rotation about the \( x \)- and \( y \)-axes are the same for \( b = \frac{5}{2} \).

(c) We use cylindrical shells to calculate the volume of rotation of \( \mathcal{R}_2 \) about the \( x \)-axis:
\[
R_{2,x} = 2 \pi \int_0^{b^2} y \,(\sqrt{y}) \, dy = 2 \pi \left[ \frac{2}{5} y^{5/2} \right]_0^{b^2} = \frac{4}{5} \pi b^5.
\]
We already know the volume of rotation of \( \mathcal{R}_1 \) about the \( x \)-axis from part (b), and \( R_{1,x} = R_{2,x} \) \( \iff \) \( \frac{1}{5} \pi b^5 = \frac{4}{5} \pi b^5 \), which has no solution for \( b \neq 0 \).

(d) We use disks to calculate the volume of rotation of \( \mathcal{R}_2 \) about the \( y \)-axis: \( R_{2,y} = \pi \int_0^{b^2} \,(\sqrt{y}) \, dy = \pi \left[ \frac{2}{3} y^{3/2} \right]_0^{b^2} = \frac{1}{2} \pi b^4 \).

We know the volume of rotation of \( \mathcal{R}_1 \) about the \( y \)-axis from part (b), and \( R_{1,y} = R_{2,y} \) \( \iff \) \( \frac{1}{2} \pi b^4 = \frac{1}{2} \pi b^4 \). But this equation is true for all \( b \), so the volumes of rotation about the \( y \)-axis are equal for all values of \( b \).